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EXACT SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS THAT DETERMINE NULL FIELDS

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ABSTRACT

The role of the null electromagnetic field, within the framework of classical relativistic electrodynamics, has never been fully established. The object of this dissertation has been to present a large class of solutions to the Einstein-Maxwell null field equations. It has been found that the nature of the solutions depends on the form of a fundamental null vector. In the cases for which the divergence of this vector vanishes, an extensive class of solutions has been obtained. The alternate case, when the divergence is non-zero, has not been fully investigated. In the latter case, it has been possible to obtain a particular solution only.

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CHAPTER I

Introduction

In the latter half of the nineteenth century Clerk Maxwell formulated a theory of electrodynamics, the first mathematically consistent theory to successfully describe electromagnetic phenomena in terms of fields. Although Michael Faraday introduced the field-concept into electrodynamics about thirty years before the appearance of Maxwell's work, he was unable to provide a complete mathematical theory for the description of his empirical results. Applying Faraday's ideas, Maxwell produced such a theory, a theory that is still the accepted formulation of classical electrodynamics.

Maxwell's theory is a macroscopic theory, describing the electrodynamic properties of ponderable bodies. The success of this work prompted Lorentz [1] to suggest a generalization which attempted to describe the behavior of electrons. The Lorentz theory was based on the Newtonian concepts of space, time, mechanics and gravitation and in addition the following assumptions were made.

(i) There exists a reference frame which is in a state of absolute rest.

(ii) All electrodynamic phenomena are determined by two fundamental entities, the electric field $\vec{E}$ and the magnetic field $\vec{H}$, quantities that are defined in the above frame of reference by

$$\vec{F} = \rho(\vec{E} + \vec{u} \times \vec{H}) \quad , \tag{1.1}$$

$\vec{F}$ being the force exerted on a unit volume having charge density ρ and velocity $\vec{u}$.

(iii) The field equations satisfied by $\vec{E}$ and $\vec{H}$ are

$$\operatorname{div} \vec{H} = 0 , \tag{1.2}$$

$$\operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} , \tag{1.3}$$

$$\operatorname{div} \vec{E} = \rho , \tag{1.4}$$

$$\operatorname{curl} \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{\rho \vec{u}}{c} , \tag{1.5}$$

where c denotes the velocity of light.

The above formulation was not invariant to coordinate transformations determined by frames of reference which were not at rest. It was only in frames of reference at absolute rest that equations (1.1) - (1.5) were assumed to be true. For example, suppose that X is a rectangular cartesian coordinate system which is at rest and X' is a similar frame which was coincident with X at time $t = 0$ but is moving in the positive x-direction at a constant velocity v. The Newtonian concept of kinematics implied

$$\begin{aligned} x' &= x - vt , \\ y' &= y , \\ z' &= z , \\ t' &= t . \end{aligned} \tag{1.6}$$

A simple calculation shows that equations (1.1) - (1.5) do not hold in reference frame X'.

Lorentz [2] was aware of this implication, and proved that the field equations were invariant to transformations of the form

$$
\begin{aligned}
x' &= \beta(x - vt) \ , \\
y' &= y \ , \\
z' &= z \ , \\
t' &= \beta\left(t - \frac{vx}{c^2}\right) \ ,
\end{aligned}
\tag{1.7}
$$

where

$$\beta = \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{1.8}$$

In the new coordinate system, the field quantities $\vec{E}'$, $\vec{H}'$ and ρ' are given by

$$E'_x = E_x \ , \quad E'_y = \beta(E_y - v/c\ H_z) \ , \quad E'_z = \beta(E_z + v/c\ H_y) \ , \tag{1.9}$$

$$H'_x = H_x \ , \quad H'_y = \beta(H_y + v/c\ E_z) \ , \quad H'_z = \beta(H_z - v/c\ E_y) \ , \tag{1.10}$$

$$\rho' = \rho\ \beta\left(1 - \frac{u_x v}{c^2}\right) \ . \tag{1.11}$$

Lorentz did not see the implications that his work might have on the fundamental concepts of space and time. Such implications were introduced by Einstein [3], and incorporated into his special theory of relativity.

By modifying some of the fundamental assumptions of kinematics, Einstein was able to show that equations (1.7) are indeed an acceptable set of transformation equations determined by X and X'. Furthermore the principle of relativity implied that $\vec{E}'$, $\vec{H}'$ and ρ' are the corresponding electrodynamic quantities as measured in the moving frame. An important consequence of this conclusion is that there is no real distinction between electric and magnetic fields. A field which seems to be purely electric in one frame could appear to be partially magnetic in another.

Although the special theory was to some extent obtained through electromagnetic considerations, it soon became evident that the theory implied other fundamental changes. For example, the theory predicted that mass and energy are equivalent entities. From this equivalence, one might suspect that the energy of the electromagnetic field, and hence the field itself would be influenced by the presence of gravitating matter.

For some years the Newtonian theory of gravitation and the Maxwell-Lorentz theory of electrodynamics existed as two disjoint theories designed to describe what was believed to be independent physical phenomena. The special theory of relativity gave the first indication that gravitational and electromagnetic fields might be two seemingly different manifestations of one and the same field. Einstein wholeheartedly accepted this view and spent almost forty years in an attempt to design a unified field theory which would describe both gravitational and electromagnetic phenomena.

In 1908 Minkowski [4] expressed the concepts of special relativity in terms of a four dimensional pseudo-Euclidean space, a formulation that led

Einstein to the discovery of the general theory of relativity. In the Minkowski formulation, a physical event is considered to be a point in a complex four-space with coordinates (x, y, z, ict). The history of a particle is described by a curve in this space-time.

Minkowski referred to these curves as world-lines and felt that the ultimate formulation of dynamics would be in terms of such paths. Unfortunately his untimely death prevented him from exploiting his ideas and the work of Minkowski remained as an equivalent formulation of the special theory of relativity.

Einstein [5] attempted to geometricize physics in the sense that physical phenomena would influence and be detected by the geometrical structure of the space-time continuum. To accomplish this purpose he assumed:

(i) the laws of physics are the same for all observers, i.e., the equations which express physical laws are form invariant (covariant) under the general group of four-dimensional coordinate transformations,

(ii) space-time is a four dimensional Riemannian manifold with signature ± 2; an event is a point in such a space and the distance ds between two events is given by

$$ds^2 = g_{ij} \, dx^i \, dx^j \quad , \tag{1.12}$$

where the g_{ij} ; $i,j = 1, 2, 3, 4$; are functions of the real coordinates x^i ; $i = 1, 2, 3, 4$.

Some years earlier, Ricci and Levi-Civita had developed a branch of analysis called the calculus of tensors which was designed to generate covariant expressions. Einstein found this theory to be ideal for the mathematical formulation of his physical ideas. The set of functions g_{ij} are the components of the so-called metric tensor, a tensor that characterizes the structure of space-time.

Einstein assumed that the path of a free particle in a purely gravitational field would be a geodesic in space-time. In addition to its geometrical interpretation, the metric tensor acts as a generalization of the Newtonian gravitational potential. To obtain a complete theory of gravitation, Einstein was faced with the problem of determining a set of field equations, which, together with boundary conditions, would determine the components of the metric tensor. Using the classical gravitational equation of Poisson as a guide, Einstein required that the field equations be second order partial differential equations, linear in the second derivatives of g_{ij}. It can be shown that the simplest tensor which linearly involves the second derivatives of the g_{ij} is the Ricci tensor, R_{ij}, a tensor given by

$$R_{ij} \equiv (\log\sqrt{-g})_{,ij} - \left\{{\alpha \atop ij}\right\}_{,\alpha} + \left\{{\alpha \atop i\beta}\right\}\left\{{\beta \atop \alpha j}\right\} - \left\{{\alpha \atop ij}\right\}(\log\sqrt{-g})_{,\alpha} \quad , (1.13)$$

where

$$(\quad)_{,i} \equiv \frac{\partial}{\partial x^i}(\quad) \quad , \qquad (1.14)$$

$$\left\{{i \atop jk}\right\} \equiv \tfrac{1}{2}\, g^{i\alpha}(g_{k\alpha,j} + g_{\alpha j,k} - g_{jk,\alpha}) \quad , \qquad (1.15)$$

and g is the determinant of g_{ij}. Einstein therefore considered the

relativistic analogue of Laplace's equation to be $R_{ij} = 0$ and assumed these equations to be the field equations for empty space. Inside matter, he assumed, as field equations,

$$R_{ij} - \frac{1}{2} g_{ij} R = - k T_{ij} \quad , \tag{1.16}$$

where $R = g^{ij} R_{ij}$, k is a constant, and T_{ij} is the so-called energy-momentum tensor. It was assumed that the energy-momentum tensor is determined by the physical properties of gravitational matter. In fact, T_{ij} is defined in a Minkowski reference frame (i.e. one in which g_{ij} has the diagonal form [-1, -1, -1, 1]) by

$$T_{ij} \equiv \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} & cg_x \\ p_{yx} & p_{yy} & p_{yz} & cg_y \\ p_{zx} & p_{zy} & p_{zz} & cg_z \\ cg_x & cg_y & cg_z & c^2\rho \end{pmatrix} \quad , \tag{1.17}$$

where the p's are the components of absolute stress, $\vec{g}$ is the momentum density and ρ is the mass density. The conservation laws for mass and momentum are expressed by the relation

$$T^{i}{}_{j;i} = 0 \quad , \tag{1.18}$$

where the semi-colon denotes covariant differentiation with respect to the Christoffel symbols.

Electromagnetic fields were incorporated into the theory by means of the energy-momentum tensor and Einstein was able to show that a suitable expression for this tensor was

$$(T_{ij})_{em} \equiv - F_{i\alpha}F_j^{\ \alpha} + \tfrac{1}{4}\, g_{ij}\, F_{\alpha\beta}\, F^{\alpha\beta} \ , \tag{1.19}$$

where F_{ij} is the electromagnetic tensor. In a Minkowski frame of reference, F_{ij} is given by

$$F_{ij} = \begin{pmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \ , \tag{1.20}$$

For source free space, the field equations were assumed to be

$$R_{ij} - \tfrac{1}{2}\, g_{ij}\, R = k(F_{i\alpha}F_j^{\ \alpha} - \tfrac{1}{4}\, g_{ij}\, F_{\alpha\beta}\, F^{\alpha\beta}) \ , \tag{1.21}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \ , \tag{1.22}$$

$$F^{ij}_{\ \ ;j} = 0 \ , \tag{1.23}$$

where (1.22) and (1.23) are the covariant versions of the Maxwell-Lorentz free-space equations.

Einstein considered the above formulation to be an interaction theory, not a unified field theory. Indeed, he went on to study many generalizations of his 1915 theory which he hoped would lead to the complete unification of the electromagnetic and gravitational fields.

In 1924, Rainich [6] provided an interesting variation of the above mentioned theory. Rainich showed that the Einstein-Maxwell equations were equivalent to

$$R = 0 \ , \tag{1.24}$$

$$R_{i\alpha} R^{\alpha j} = \tfrac{1}{4} \delta^j_i R_{\alpha\beta} R^{\alpha\beta} \ , \tag{1.25}$$

$$\alpha_{i,j} - \alpha_{j,i} = 0 \ , \tag{1.26}$$

where

$$\alpha^i \equiv - \frac{e^{ijk\ell} R_{j\alpha;k} R^\alpha_\ell}{\sqrt{-g}\, R_{\alpha\beta} R^{\alpha\beta}} \ , \tag{1.27}$$

provided that the electromagnetic field is non-null. The Rainich result, later duplicated by Misner and Wheeler [7], shows that the general theory might be regarded as a true unified field theory, for all fields are now determined by the metric tensor.

An electromagnetic field is null if and only if

$$F_{ij} F^{ij} = 0 \ , \quad e^{ijk\ell} F_{ij} F_{k\ell} = 0 \ . \tag{1.28}$$

For such fields it can be shown that

$$R_{\alpha\beta} R^{\alpha\beta} = 0 \quad , \tag{1.29}$$

$$R_{j\alpha;k} R^{\alpha}{}_{\ell} = 0 \quad . \tag{1.30}$$

Hence, the α^i as defined in (1.27) is now indeterminate and the field equations become meaningless.

The physical significance of null fields has not yet been established. Rainich believed that the electromagnetic field tensor should resemble an analytic function. In that case, the vanishing of the invariants (1.28) in one region of space would imply that they vanish everywhere. Since there are regions where these invariants are non-zero, then they would have to be non-zero everywhere except for isolated points. Hence, Rainich felt that null-fields do not exist. The fact that the Rainich-Misner-Wheeler description of electromagnetism broke down for null-fields gave rise to a conjecture by Witten [8] that non-trivial null-fields might be ruled out by the Einstein-Maxwell equations. However, Hlavaty [9] demonstrated that all source-free fields can be geometricized and has produced examples of such structures.

The object of the present thesis is the mathematical determination of null-field solutions to the source-free relativistic electromagnetic field equations, a problem that has been considered under specialized conditions by other authors. Our procedure will show that the general situation can be obtained by the separate discussion of three cases.

For the first case, the general solution has been provided. The second and third cases are more involved and the general solution has not been obtained. However, even for these cases, it has been possible to exhibit an extensive class of solutions.

CHAPTER II

Preliminary Results

By using a suitable choice of units it can be shown that the Einstein-Maxwell field equations may be put in the form

$$R_{ij} = F_{i\alpha} F_j{}^{\alpha} + {}^*F_{i\alpha} {}^*F_j{}^{\alpha} \quad , \tag{2.1}$$

$$F^i{}_{j;i} = 0 \quad , \tag{2.2}$$

$${}^*F^i{}_{j;i} = 0 \quad , \tag{2.3}$$

where

$${}^*F_{ij} \equiv \tfrac{1}{2}\sqrt{-g}\; e_{ijk\ell}\, F^{k\ell} \quad . \tag{2.4}$$

When F_{ij} is a null field, it has been shown (see [7]) that

$$F_{ij} = w_i e_j - w_j e_i \quad , \tag{2.5}$$

$${}^*F_{ij} = w_i h_j - w_j h_i \quad , \tag{2.6}$$

$$R_{ij} = 2w_i w_j \quad , \tag{2.7}$$

where w_i is a null vector; e_i and h_i are unit vectors; and w_i, e_i, h_i are orthogonal to each other. Although the vector w_i is uniquely determined (except for its algebraic sign), the vectors e_i and h_i are determined only up to additive multiples of w_i. For every pair

of scalars (ξ,η) , the replacement of e_i and h_i in (2.5) and (2.6) by e'_i and h'_i given by

$$e'_i = e_i + \xi w_i \quad , \tag{2.8}$$

$$h'_i = h_i + \eta w_i \quad , \tag{2.9}$$

leaves the electromagnetic field tensors F_{ij} and $*F_{ij}$ unchanged.

If P is an arbitrary point in space-time, then there exists a system of coordinates such that the metric tensor g_{ij} and the vectors w_i , e_i , h_i , have the following forms at P:

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{2.10}$$

$$w_i = (1, 0, 0, 0) \quad , \tag{2.11}$$

$$e_i = (0, 0, 1, 0) \quad , \tag{2.12}$$

$$h_i = (0, 0, 0, 1) \quad . \tag{2.13}$$

If a_i is a vector defined by

$$\begin{aligned} a_i w^i &= 1 \ , & a_i a^i &= 0 \ , \\ a_i e^i &= 0 \ , & a_i h^i &= 0 \ , \end{aligned} \tag{2.14}$$

then at P

$$a_i = (0, 1, 0, 0) \quad . \tag{2.15}$$

From (2.10), it follows that a covariant form of the metric is given by

$$g_{ij} = w_i a_j + w_j a_i + e_i e_j + h_i h_j \quad . \tag{2.16}$$

Hence, a determination of the vectors w_i, a_i, e_i and h_i is equivalent to a determination of the metric tensor.

From (2.2), (2.3), (2.7), and

$$g_{ij;k} = 0 \quad , \tag{2.17}$$

$$R^{ij}{}_{;j} = 0 \quad , \tag{2.18}$$

it is possible to obtain

$$w_{i;j} e^j + w_i e^j{}_{;j} - e_{i;j} w^j - e_i w^j{}_{;j} = 0 \quad , \tag{2.2a}$$

$$w_{i;j} h^j + w_i h^j{}_{;j} - h_{i;j} w^j - h_i w^j{}_{;j} = 0 \quad , \tag{2.3a}$$

$$\begin{aligned} &w^i w_{i;j} = 0 \quad , && e^i e_{i;j} = 0 \quad , \\ &h^i h_{i;j} = 0 \quad , && w^i e_{i;j} + e_i w^i{}_{;j} = 0 \quad , \\ &w^i h_{i;j} + h_i w^i{}_{;j} = 0 \quad , && h^i e_{i;j} + e_i h^i{}_{;j} = 0 \quad , \end{aligned} \tag{2.17a}$$

$$w^{j}{}_{;j}\, w_i + w^j\, w_{i;j} = 0 \ . \tag{2.18a}$$

By evaluating at the point P , one obtains

$$w_{i;j} = \begin{pmatrix} w_{1;1} & -w^{i}{}_{;i} & w_{1;3} & w_{1;4} \\ 0 & 0 & 0 & 0 \\ w_{3;1} & 0 & w^{i}{}_{;i} & w_{3;4} \\ w_{4;1} & 0 & -w_{3;4} & w^{i}{}_{;i} \end{pmatrix} \ . \tag{2.19}$$

The characteristic equation of the symmetric tensor

$$w_{ij} \equiv \tfrac{1}{2}\,(w_{i;j} + w_{j;i}) \quad , \tag{2.20}$$

is

$$|w_{ij} - \lambda\, g_{ij}| = \begin{vmatrix} w_{11} & -\frac{1}{2}w^{i}{}_{;i} - \lambda & w_{13} & w_{14} \\ -\frac{1}{2}w^{i}{}_{;i} - \lambda & 0 & 0 & 0 \\ w_{13} & 0 & w^{i}{}_{;i} - \lambda & 0 \\ w_{14} & 0 & 0 & w^{i}{}_{;i} - \lambda \end{vmatrix} = 0 \ . \tag{2.21}$$

Hence, w_{ij} has the two real eigen values

$$\lambda_1 = -\tfrac{1}{2}\, w^{i}{}_{;i} \quad , \qquad \lambda_2 = w^{i}{}_{;i} \ . \tag{2.22}$$

Further, $\lambda_1 \neq \lambda_2$ unless $w^i{}_{;i} = 0$. Equation (2.21) enables us to enumerate the various possibilities for the elementary divisors of $|w_{ij} - \lambda\, g_{ij}|$. By Eisenhart [11], one can classify the reductions of the tensors w_{ij} and g_{ij}. These reductions lead to the following three situations:

<u>Case I.</u> The condition

$$w_{ij}\, w^{ik} = 0 \;, \tag{2.23}$$

implies

$$w^i{}_{;i} = w_{13} = w_{14} = 0 \;. \tag{2.23a}$$

and the elementary divisors of (2.21) are then

$$\lambda^2 \,,\; \lambda \,,\; \lambda \;.$$

Eisenhart states that there is a coordinate system in which

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix} \;, \tag{2.24}$$

$$w_{ij} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;, \tag{2.25}$$

at the point P.

Case II. If

$$w_{ij} w^{jk} \neq 0 \;, \tag{2.26}$$

and

$$w^{i}{}_{;i} = 0 \;, \tag{2.27}$$

then the elementary divisors are

$$\lambda^3 \;, \; \lambda \;.$$

The metric tensor is as given in (2.24) and w_{ij} now has the form:

$$w_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;. \tag{2.28}$$

Case III. Finally, if

$$w^{i}{}_{;i} = -\alpha_2 \neq 0 \;, \tag{2.29}$$

the elementary divisors are

$$(\lambda - \tfrac{1}{2}\alpha_2)^2 \;, \quad (\lambda + \alpha_2) \;, \quad (\lambda + \alpha_2) \;.$$

Again, the metric is as given in (2.24) and w_{ij} is given by:

$$w_{ij} = \begin{pmatrix} \alpha_1 & \frac{1}{2}\alpha_2 & 0 & 0 \\ \frac{1}{2}\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & -k_3\alpha_2 & 0 \\ 0 & 0 & 0 & -k_4\alpha_2 \end{pmatrix} . \qquad (2.30)$$

A detailed discussion of each of the above cases is given in subsequent chapters.

CHAPTER III

Discussion of Case I

If a coordinate system is chosen such that

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.1}$$

$$\begin{aligned} w_i &= (1, 0, 0, 0) , \quad a_i = (0, 1, 0, 0) , \\ e_i &= (0, 0, 1, 0) , \quad h_i = (0, 0, 0, 1) , \end{aligned} \tag{3.2}$$

then the field equations, together with the condition

$$w_{ij} \, w^{jk} = 0 , \tag{3.3}$$

imply

$$w_{ij} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.4}$$

Hence,

$$w_{i;j} = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ -\alpha_3 & 0 & 0 & \alpha_5 \\ -\alpha_4 & 0 & -\alpha_5 & 0 \end{pmatrix} , \tag{3.5}$$

where the α's may be regarded as unknown invariants evaluated at P. Similarly, equations (2.2a), (2.3a), (2.17a) and (2.14) may be used to obtain

$$e_{i;j} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \alpha_3 & 0 & 0 & -\alpha_5 \\ 0 & 0 & 0 & 0 \\ \beta_5 & -\alpha_5 & 2\alpha_4 & -2\alpha_3 \end{pmatrix} , \qquad (3.6)$$

$$h_{i;j} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \alpha_4 & 0 & \alpha_5 & 0 \\ -\beta_5 & \alpha_5 & -2\alpha_4 & 2\alpha_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad (3.7)$$

$$a_{i;j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -\alpha_3 & -\alpha_4 \\ -\beta_1 & -\beta_2 & -\beta_3 & -\beta_4 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \end{pmatrix} . \qquad (3.8)$$

If one introduces the vectors

$$\alpha_i \equiv \alpha_1 w_i + \alpha_3 e_i + \alpha_4 h_i , \qquad (3.9)$$

$$\beta_i \equiv \beta_1 w_i + \beta_2 a_i + \beta_3 e_i + \beta_4 h_i , \qquad (3.10)$$

$$\gamma_i \equiv \gamma_1 w_i + \gamma_2 a_i + \gamma_3 e_i + \gamma_4 h_i , \qquad (3.11)$$

$$\xi_i \equiv - \alpha_3 w_i + \alpha_5 h_i , \qquad (3.12)$$

$$\eta_i \equiv - \alpha_4 w_i - \alpha_5 e_i \quad , \tag{3.13}$$

$$\zeta_i \equiv \beta_5 w_i - \alpha_5 a_i + 2\alpha_4 e_i - 2\alpha_3 h_i \quad , \tag{3.14}$$

then equations (3.5) - (3.8) may be expressed covariantly as

$$w_{i;j} = w_i \alpha_j + e_i \xi_j + h_i \eta_j \quad , \tag{3.15}$$

$$e_{i;j} = w_i \beta_j - a_i \xi_j + h_i \zeta_j \quad , \tag{3.16}$$

$$h_{i;j} = w_i \gamma_j - a_i \eta_j - e_i \zeta_j \quad , \tag{3.17}$$

$$a_{i;j} = - a_i \alpha_j - e_i \beta_j - h_i \gamma_j \quad . \tag{3.18}$$

The integrability conditions for equations (3.15) - (3.18) are

$$w_{i;jk} - w_{i;kj} = w^\alpha R_{\alpha ijk} \quad , \tag{3.19}$$

$$a_{i;jk} - a_{i;kj} = a^\alpha R_{\alpha ijk} \quad , \tag{3.20}$$

$$e_{i;jk} - e_{i;kj} = e^\alpha R_{\alpha ijk} \quad , \tag{3.21}$$

$$h_{i;jk} - h_{i,kj} = h^\alpha R_{\alpha ijk} \quad . \tag{3.22}$$

Equations (3.19) - (3.22) are sufficient to specify $R_{ijk\ell}$ at the point P. Since the components of $R_{ijk\ell}$ involve the α's , β's and γ's , the identities

$$R_{ijk\ell} = - R_{jik\ell} = - R_{ij\ell k} , \tag{3.23}$$

$$R_{ijk\ell} = R_{k\ell ij} , \tag{3.24}$$

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0 , \tag{3.25}$$

and the field equation

$$R^{\alpha}_{\ ij\alpha} = 2w_i \, w_j , \tag{3.26}$$

produce a set of first order partial differential equations which may be used to determine the unknown invariants. The computations leading to the $R_{ijk\ell}$ are straightforward but space-consuming. For convenience, these calculations are omitted. The appendix contains a list of the non-zero components of the curvature tensor.

The set of equations

$$R_{22} = R_{3223} + R_{4224} = 0 , \tag{3.27}$$

$$R_{23} = R_{1232} + R_{4234} = 0 , \tag{3.28}$$

$$R_{24} = R_{1242} + R_{3243} = 0 , \tag{3.29}$$

$$R_{33} = 2 \, R_{2331} + R_{4334} = 0 , \tag{3.30}$$

$$R_{34} = R_{2341} + R_{2431} = 0 , \tag{3.31}$$

$$R_{44} = 2\, R_{2441} + R_{3443} = 0 \quad , \tag{3.32}$$

$$R_{2134} + R_{2341} + R_{2413} = 0 \quad ,$$

give rise to the set

$$\alpha_5 = 0 \quad , \tag{3.33}$$

$$\alpha_{3,2} = 0 \quad , \qquad \alpha_{4,2} = 0 \quad , \tag{3.34}$$

$$\alpha_{3,3} = 3\, \alpha_4^{\,2} \quad , \qquad \alpha_{4,3} = -3\, \alpha_3\, \alpha_4 \quad , \tag{3.35}$$

$$\alpha_{3,4} = -3\, \alpha_3\, \alpha_4 \quad , \qquad \alpha_{4,4} = 3\, \alpha_3^{\,2} \quad , \tag{3.36}$$

equations that hold at the point P. This latter restriction can be relaxed by writing, for example,

$$\alpha_{3,i} = \varphi\, w_i + 3\, \alpha_4^{\,2}\, e_i - 3\, \alpha_3\, \alpha_4\, h_i \quad , \tag{3.37}$$

where φ is undetermined. The integrability condition for (3.37) is

$$\alpha_{3,i;j} - \alpha_{3,j;i} = 0 \quad , \tag{3.38}$$

a set of equations that yields

$$\alpha_3 = \alpha_4 = 0 \quad . \tag{3.39}$$

Hence, (3.15) becomes

$$w_{i;j} = \alpha_1\, w_i\, w_j \quad . \tag{3.40}$$

The fact that $w_{i;j}$ is a symmetric tensor implies that w_i is a gradient, i.e. we may put

$$w_i = w_{,i} \quad , \tag{3.41}$$

where w is some invariant.

It is well known (see Eisenhart [12]) that there exists a coordinate system in which w^i has the form

$$w^i = (0,\ w^2,\ 0,\ 0) \quad , \tag{3.42}$$

at all points in the space. Since w_i is a null vector then

$$w_{,2} = 0 \quad , \tag{3.43}$$

in this system of coordinates. To avoid a non-trivial solution, it is necessary to assume that w is a function of at least one of x^1, x^3, x^4. There is no loss of generality in assuming that w is a function of x^1. This being the case, the transformation

$$\bar{x}^1 = w \quad , \quad \bar{x}^2 = x^2 \quad , \quad \bar{x}^3 = x^3 \quad , \quad \bar{x}^4 = x^4 \quad , \tag{3.44}$$

has a non-zero Jacobian and in the new coordinate system

$$w_i = (1,\ 0,\ 0,\ 0) \quad , \tag{3.45}$$

while equation (3.42) remains unchanged.

Since w^i is orthogonal to e_i , then

$$e_i = (e_1,\ 0,\ e_3,\ e_4) \quad . \tag{3.46}$$

Also, (3.16) implies

$$e_{i,j} - e_{j,i} = e_{i;j} - e_{j;i} = \beta_2(w_i a_j - w_j a_i)$$
$$+ \beta_3(w_i e_j - w_j e_i) + (\beta_4 - \beta_5)(w_i h_j - w_j h_i) \ , \tag{3.47}$$

from which one obtains

$$e_{3,2} = 0 \ , \quad e_{4,2} = 0 \ , \quad e_{3,4} - e_{4,3} = 0 \ . \tag{3.48}$$

The general solution to (3.48) is

$$e_3 = \mu_{,3} \ , \quad e_4 = \mu_{,4} \ , \tag{3.49}$$

where μ can be chosen to be independent of x^2. Hence

$$e_i = \mu_{,i} + \xi w_i \ , \tag{3.50}$$

and since F_{ij} is independent of the value of ξ, we may choose $\xi = 0$.

Similarly, we may put

$$h_i = \nu_{,i} \ , \tag{3.51}$$

where ν is independent of x^2. Since e_i and h_i are unit vectors, then μ and ν must be functions of x^3 and x^4. If μ is assumed to be a function of x^3, then the fact that e_i is orthogonal to h_i implies that ν must be a function of x^4. Hence the transformation

$$\bar{x}^1 = x^1 \ , \quad \bar{x}^2 = x^2 \ , \quad \bar{x}^3 = \mu \ , \quad \bar{x}^4 = \nu \ , \tag{3.52}$$

is admissible and in the new system of coordinates we will have

$$e_i = (0, 0, 1, 0) \ , \quad h_i = (0, 0, 0, 1) \ . \tag{3.53}$$

Finally, if we put

$$\bar{x}^2 = \int \frac{dx^2}{w^2} , \tag{3.54}$$

the form of w_i , e_i and h_i will be unaffected but w^i reduces to

$$w^i = (0, 1, 0, 0) . \tag{3.55}$$

However, a_i will now have the form

$$a_i = (\alpha, 1, \beta, \gamma) , \tag{3.56}$$

where α, β and γ are unknown functions.

Equation (2.16) determines the form of the metric tensor. Hence ,

$$g_{ij} = \begin{pmatrix} 2\alpha & 1 & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix} , \tag{3.57}$$

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \beta^2+\gamma^2-2\alpha & -\beta & -\gamma \\ 0 & -\beta & 1 & 0 \\ 0 & -\gamma & 0 & 1 \end{pmatrix} , \tag{3.58}$$

$$\sqrt{-g} = 1 . \tag{3.59}$$

Equations (3.40) and (3.45) imply

$$\left\{ {1 \atop ij} \right\} = 0 \;, \qquad i \neq 1 \neq j \;. \tag{3.60}$$

This equation is satisfied identically except when i equals one and j is three or four. These two cases imply

$$\beta_{,2} = 0 \;, \qquad \gamma_{,2} = 0 \;. \tag{3.61}$$

The non-zero Christoffel symbols are:

$$\left.\begin{aligned}
&\left\{ {1 \atop 11} \right\} = -\,\alpha_{,2} \;, \qquad \left\{ {2 \atop 12} \right\} = \alpha_{,2} \;, \\
&\left\{ {2 \atop 33} \right\} = \beta_{,3} \;, \qquad \left\{ {2 \atop 44} \right\} = \gamma_{,4} \;, \\
&\qquad \left\{ {2 \atop 34} \right\} = \tfrac{1}{2}(\gamma_{,3} + \beta_{,4}) \;, \\
&\left\{ {3 \atop 14} \right\} = -\left\{ {4 \atop 13} \right\} = \tfrac{1}{2}\,(\beta_{,4} - \gamma_{,3}) \;, \\
&\left\{ {2 \atop 13} \right\} = \tfrac{1}{2}\,[2\alpha_{,3} - \gamma(\gamma_{,3} - \beta_{,4})] \;, \\
&\left\{ {2 \atop 14} \right\} = \tfrac{1}{2}\,[2\alpha_{,4} - \beta(\beta_{,4} - \gamma_{,3})] \;, \\
&\left\{ {3 \atop 11} \right\} = \beta_{,1} + \beta\,\alpha_{,2} - \alpha_{,3} \;, \\
&\left\{ {4 \atop 11} \right\} = \gamma_{,1} + \gamma\,\alpha_{,2} - \alpha_{,4} \;, \\
&\left\{ {2 \atop 11} \right\} = \alpha_{,1} + (2\alpha - \beta^2 - \gamma^2)\,\alpha_{,2} + \beta\,\alpha_{,3} \\
&\qquad + \gamma\,\alpha_{,4} - \beta\,\beta_{,1} - \gamma\,\gamma_{,1} \;.
\end{aligned}\right\} \tag{3.62}$$

According to (3.45), (3.59) and (1.13), the field equations (2.7) reduce to

$$- \left\{ {\alpha \atop ij} \right\}_{,\alpha} + \left\{ {\alpha \atop i\beta} \right\} \left\{ {\beta \atop \alpha j} \right\} = 2\, \delta^1_i\, \delta^1_j \quad . \tag{3.63}$$

In turn these equations yield

$$\begin{aligned} R_{11} &= \nabla^2 \alpha - (2\alpha - \beta^2 - \gamma^2)\, \alpha_{,22} - \alpha_{,2}(\beta_{,3} + \gamma_{,4}) \quad , \\ &\quad - \tfrac{1}{2}(\beta_{,4} - \gamma_{,3})^2 - 2\beta\, \alpha_{,23} - 2\gamma\, \alpha_{,24} \\ &\quad - (\beta_{,3} + \gamma_{,4})_{,1} = 2 \quad , \end{aligned} \tag{3.64}$$

$$R_{12} = - \alpha_{,22} = 0 \quad , \tag{3.65}$$

$$R_{13} = - \alpha_{,32} - \tfrac{1}{2}(\gamma_{,3} - \beta_{,4})_{,4} = 0 \quad , \tag{3.66}$$

$$R_{14} = - \alpha_{,42} - \tfrac{1}{2}(\beta_{,4} - \gamma_{,3})_{,3} = 0 \quad , \tag{3.67}$$

where

$$\nabla^2(\quad) \equiv (\quad)_{,33} + (\quad)_{,44} \quad . \tag{3.68}$$

The general solution of (3.65), (3.66), and (3.67) can be taken to be

$$\alpha_{,2} = - \bar{\gamma}_{,4} \quad , \qquad \tfrac{1}{2}(\gamma_{,3} - \beta_{,4}) = \bar{\gamma}_{,3} \quad , \tag{3.69}$$

$$\alpha_{,2} = - \bar{\beta}_{,3} \quad , \qquad \tfrac{1}{2}(\gamma_{,3} - \beta_{,4}) = - \bar{\beta}_{,4} \quad , \tag{3.70}$$

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$$\bar{\beta}_{,2} = 0 \;, \qquad \bar{\gamma}_{,2} = 0 \;. \tag{3.71}$$

If the last equations of (3.69) and (3.70) are added one obtains

$$(\gamma - \bar{\gamma})_{,3} - (\beta - \bar{\beta})_{,4} = 0 \;. \tag{3.72}$$

Hence, there exists a function $\theta = \theta(x^1, x^3, x^4)$ such that

$$\beta - \bar{\beta} = \theta_{,3} \;, \qquad \gamma - \bar{\gamma} = \theta_{,4} \;. \tag{3.73}$$

The coordinate transformation

$$\bar{x}^1 = x^1 \;, \quad \bar{x}^2 = x^2 + \theta \;, \quad \bar{x}^3 = x^3 \;, \quad \bar{x}^4 = x^4 \;, \tag{3.74}$$

gives $\bar{g}_{ij}$ the form

$$\bar{g}_{ij} = \begin{pmatrix} 2\alpha & 1 & \bar{\beta} & \bar{\gamma} \\ 1 & 0 & 0 & 0 \\ \bar{\beta} & 0 & 1 & 0 \\ \bar{\gamma} & 0 & 0 & 1 \end{pmatrix} \;. \tag{3.75}$$

Dropping the bars, the new α , β and γ will satisfy

$$\begin{aligned} \beta_{,3} &= \gamma_{,4} \;, \qquad \beta_{,4} = -\gamma_{,3} \;, \\ \alpha_{,2} &= -\beta_{,3} \;. \end{aligned} \tag{3.76}$$

Equation (3.64) can now be written

$$\begin{aligned} \nabla^2 \alpha = \alpha_{,2} (\beta_{,3} + \gamma_{,4}) &+ \tfrac{1}{2}(\beta_{,4} - \gamma_{,3})^2 + 2\beta\, \alpha_{,23} \\ &+ 2\gamma\, \alpha_{,24} + (\beta_{,3} + \gamma_{,4})_{,1} + 2 \;. \end{aligned} \tag{3.77}$$

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From (3.76) and (3.77) one obtains

$$\nabla^2\alpha = 2(\gamma\,\gamma_{,3} - \beta\,\beta_{,3} + \beta_{,1} + x^3)_{,3} \ , \tag{3.78}$$

or

$$\nabla^2\alpha = 2(\beta\,\beta_{,4} - \gamma\,\gamma_{,4} + \gamma_{,1} + x^4)_{,4} \ . \tag{3.79}$$

Placing

$$U = \gamma\,\gamma_{,3} - \beta\,\beta_{,3} + \beta_{,1} + x^3 \ , \tag{3.80}$$

and

$$V = \beta\,\beta_{,4} - \gamma\,\gamma_{,4} + \gamma_{,1} + x^4 \ , \tag{3.81}$$

then (3.77) may be written

$$\nabla^2\alpha = U_{,3} + V_{,4} \ . \tag{3.82}$$

It is easily verified that U and V are harmonic conjugates and therefore satisfy the relation

$$\tfrac{1}{2}\nabla^2\,(x^3\,U + x^4\,V) = U_{,3} + V_{,4} \ . \tag{3.83}$$

Hence, the general solution to (3.82) and (3.76) is

$$\alpha = \tfrac{1}{2}\,(x^3\,U + x^4\,V) - x^2\,\beta_{,3} + \sigma \ , \tag{3.84}$$

where $\sigma = \sigma(x^1, x^3, x^4)$ and $\beta = \beta(x^1, x^3, x^4)$ satisfy

$$\nabla^2 \sigma = \nabla^2 \beta = 0 \ . \tag{3.85}$$

Summary of Results

If the tensor $w_{ij}\, w^{ik}$ is identically zero, then the space-time structure is as follows:

(i) there exists a coordinate system in which the components of g_{ij} are given by

$$g_{ij} = \begin{pmatrix} 2\alpha & 1 & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix} , \tag{3.57a}$$

(ii) $\beta = \beta(x^1, x^3, x^4)$ is an arbitrary solution of

$$\nabla^2 \beta = 0 , \tag{3.86}$$

and $\gamma = \gamma(x^1, x^3, x^4)$ is its harmonic conjugate ,

$$\begin{aligned} \alpha = \tfrac{1}{2} \{ (x^3)^2 + (x^4)^2 + x^3(\gamma\, \gamma_{,3} - \beta\, \beta_{,3} + \beta_{,1}) \\ + x^4(\beta\beta_{,4} - \gamma\gamma_{,4} + \gamma_{,1})\} - x^2\, \beta_{,3} + \sigma \ , \end{aligned} \tag{3.87}$$

where $\sigma = \sigma(x^1, x^3, x^4)$ is an arbitrary solution of

$$\nabla^2 \alpha = 0 \ . \tag{3.88}$$

Geometric Implications

The equations

$$R_{ij} = 2w_i w_j \quad , \tag{3.26a}$$

$$w_{i;j} = \alpha_1 w_i w_j \quad , \tag{3.40a}$$

imply

$$R = 0 \quad , \qquad R_{ij} R^{jk} = 0 \quad , \tag{3.88}$$

$$R_{ij;k} - R_{ik;j} = 0 \quad , \tag{3.89}$$

$$R_{ij;k} R^{i}{}_{\alpha;\beta} = 0 \quad . \tag{3.90}$$

It has been shown that (3.88) implies (3.26a). If P is an arbitrary point, then one can choose a coordinate system such that the point-forms of g_{ij} and w_i are as in equations (3.1) and (3.2). Since equation (3.89) may be written

$$w_i(w_{j;k} - w_{k;j}) = w_k w_{i;j} - w_j w_{i;k} \quad , \tag{3.91}$$

it follows that

$$w_{i;j} = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ 2\alpha_3 & 0 & 0 & 0 \\ 2\alpha_4 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{3.92}$$

Equation (3.90) reduces to

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$$w^{i}{}_{;\beta}\, w_{i;k} = 0 \quad , \tag{3.93}$$

and $\beta = k = 1$ yields

$$\alpha_3{}^2 + \alpha_4{}^2 = 0 \quad . \tag{3.94}$$

Hence, equations (3.88) - (3.90) are equivalent to (3.26a) and (3.40a). Since it has been shown that equation (3.40a) is sufficient to guarantee the existence of a solution, one can regard (3.88) - (3.90) as a set of necessary and sufficient conditions that must be imposed on the curvature tensor if the space-time structure is to be consistent with the restriction $w_{ij}\, w^{jk} = 0$.

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CHAPTER IV

Discussion of Case II

It follows from chapter II that this case is characterized by the following set of point equations:

$$w_{i;j} = \begin{pmatrix} \alpha_1 & 0 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ \bar{\alpha}_3 & 0 & 0 & \alpha_5 \\ \bar{\alpha}_4 & 0 & -\alpha_5 & 0 \end{pmatrix}, \tag{4.1}$$

$$\begin{aligned} w_i &= (1, 0, 0, 0) , \quad & a_i &= (0, 1, 0, 0) , \\ e_i &= (0, 0, 1, 0) , \quad & h_i &= (0, 0, 0, 1) . \end{aligned} \tag{4.2}$$

The equation

$$R_{22} = R_{3223} + R_{4224} = 0 , \tag{4.3}$$

reduces to

$$\alpha_5 = 0 . \tag{4.4}$$

But if α_5 is zero, then the covariant form of (4.1) leads to the following identity:

$$\begin{aligned} w_i(w_{j;k} - w_{k,j}) &+ w_j(w_{k;i} - w_{i;k}) \\ &+ w_k(w_{i;j} - w_{j;i}) = 0 . \end{aligned} \tag{4.5}$$

Hence, from the theory of differential equations (see Forsyth [13]) we

have that

$$w_i = e^{\tau} w_{,i} , \tag{4.6}$$

where τ and w are unknown invariants. If we choose a reference frame such that the expressions

$$w^i = (0, w^2, 0, 0) , \tag{4.7}$$

$$w_i = (e^{\tau}, 0, 0, 0) , \tag{4.8}$$

are valid at all points of the space, then Maxwell's equations imply

$$e_i = e^{-\tau} \mu_{,i} , \tag{4.9}$$

$$h_i = e^{-\tau} \nu_{,i} . \tag{4.10}$$

A repetition of the arguments used in chapter III lead us to conclude that a reference frame can be chosen such that e_i and h_i are given by

$$e_i = (0, 0, e^{-\tau}, 0) , \tag{4.11}$$

$$h_i = (0, 0, 0, e^{-\tau}) . \tag{4.12}$$

Finally, if we set

$$\bar{x}^2 = \int \frac{dx^2}{w^2} , \tag{4.13}$$

then equations (4.8), (4.11) and (4.12) will be unchanged and w^i will

simplify to

$$w^i = (0, 1, 0, 0) \quad , \tag{4.14}$$

since $w^i a_i = 1$, a_i will be of the form

$$a_i = (\alpha, 1, \beta, \gamma) \quad , \tag{4.15}$$

where α, β and γ are unknown invariants. We can now use (2.16) to obtain a general expression for the metric tensor. The results are

$$g_{ij} = \begin{pmatrix} 2\alpha & 1 & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ \beta & 0 & e^{-2\tau} & 0 \\ \gamma & 0 & 0 & e^{-2\tau} \end{pmatrix} , \tag{4.16}$$

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \sigma & -\beta e^{2\tau} & -\gamma e^{2\tau} \\ 0 & -\beta e^{2\tau} & e^{2\tau} & 0 \\ 0 & -\gamma e^{2\tau} & 0 & e^{2\tau} \end{pmatrix} , \tag{4.17}$$

where

$$\sigma \equiv -2\alpha + e^{2\tau}(\beta^2 + \gamma^2) \quad , \tag{4.18}$$

and

$$\tau_{,2} = 0 \quad . \tag{4.19}$$

The last equation follows from the condition $w^i{}_{;i} = 0$.

The non-zero Christoffel symbols are

$$\left.\begin{array}{l}
\left\{ {1 \atop 11} \right\} = -\alpha_{,2} \quad , \qquad \left\{ {1 \atop 13} \right\} = -\tfrac{1}{2}\beta_{,2} \quad , \qquad \left\{ {1 \atop 14} \right\} = -\tfrac{1}{2}\gamma_{,2} \quad , \\[2ex]
\left\{ {2 \atop 11} \right\} = \alpha_{,1} - \sigma\alpha_{,2} - \beta e^{2\tau}(\beta_{,1} - \alpha_{,3}) - \gamma e^{2\tau}(\gamma_{,1} - \alpha_{,4}) \quad , \\[2ex]
\left\{ {2 \atop 12} \right\} = \alpha_{,2} - \tfrac{1}{2} e^{2\tau}(\beta\beta_{,2} + \gamma\gamma_{,2}) \quad , \\[2ex]
\left\{ {2 \atop 13} \right\} = \alpha_{,3} - \tfrac{1}{2}\sigma\beta_{,2} + \beta\tau_{,1} - \tfrac{1}{2}\gamma e^{2\tau}(\gamma_{,3} - \beta_{,4}) \quad , \\[2ex]
\left\{ {2 \atop 14} \right\} = \alpha_{,4} - \tfrac{1}{2}\sigma\gamma_{,2} - \tfrac{1}{2}\beta e^{2\tau}(\beta_{,4} - \gamma_{,3}) + \gamma\tau_{,1} \quad , \\[2ex]
\left\{ {2 \atop 23} \right\} = \tfrac{1}{2}\beta_{,2} \quad , \qquad \left\{ {2 \atop 24} \right\} = \tfrac{1}{2}\gamma_{,2} \quad , \\[2ex]
\left\{ {2 \atop 33} \right\} = \beta_{,3} + e^{-2\tau}\tau_{,1} + \beta\tau_{,3} - \gamma\tau_{,4} \quad , \\[2ex]
\left\{ {2 \atop 34} \right\} = \tfrac{1}{2}(\gamma_{,3} + \beta_{,4}) + \beta\tau_{,4} + \gamma\tau_{,3} \quad , \\[2ex]
\left\{ {2 \atop 44} \right\} = \gamma_{,4} + e^{-2\tau}\tau_{,1} - \beta\tau_{,3} + \gamma\tau_{,4} \quad , \\[2ex]
\left\{ {3 \atop 11} \right\} = e^{2\tau}(\beta\alpha_{,2} + \beta_{,1} - \alpha_{,3}) \quad , \\[2ex]
\left\{ {3 \atop 12} \right\} = \tfrac{1}{2} e^{2\tau}\beta_{,2} \quad , \qquad \left\{ {3 \atop 13} \right\} = \tfrac{1}{2} e^{2\tau}\beta\beta_{,2} - \tau_{,1} \quad , \\[2ex]
\left\{ {3 \atop 14} \right\} = \tfrac{1}{2} e^{2\tau}(\beta\gamma_{,2} + \beta_{,4} - \gamma_{,3}) \quad ,
\end{array}\right\} \quad (4.20a)$$

$$\left.\begin{aligned}
&\{{}^{3}_{33}\} = -\tau_{,3} \;, \quad \{{}^{3}_{34}\} = -\tau_{,4} \;, \quad \{{}^{3}_{44}\} = \tau_{,3} \;, \\
&\{{}^{4}_{11}\} = e^{2\tau}(\gamma\, \alpha_{,2} + \gamma_{,1} - \alpha_{,4}) \;, \\
&\{{}^{4}_{12}\} = \tfrac{1}{2}\, e^{2\tau}\, \gamma_{,2} \;, \quad \{{}^{4}_{14}\} = \tfrac{1}{2}\, e^{2\tau}\, \gamma\gamma_{,2} - \tau_{,1} \;, \\
&\{{}^{4}_{13}\} = \tfrac{1}{2}\, e^{2\tau}(\gamma\, \beta_{,2} + \gamma_{,3} - \beta_{,4}) \;, \\
&\{{}^{4}_{33}\} = \tau_{,4} \;, \quad \{{}^{4}_{34}\} = -\tau_{,3} \;, \quad \{{}^{4}_{44}\} = -\tau_{,4} \;.
\end{aligned}\right\} \qquad (4.20b)$$

From (4.16) it follows that

$$\log \sqrt{-g} = -2\tau \;. \qquad (4.21)$$

Hence, the field equation (1.13) reduces to

$$(-2\tau)_{,ij} - \{{}^{\alpha}_{ij}\}_{,\alpha} + \{{}^{\alpha}_{i\beta}\}\{{}^{\beta}_{\alpha j}\} + 2\{{}^{\alpha}_{ij}\}\,\tau_{,\alpha} = e^{2\tau}\,\delta^{1}{}_{i}\,\delta^{1}{}_{j} \;. \qquad (4.22)$$

One can verify that

$$R_{12} = -\alpha_{,22} + \tfrac{1}{2}\, e^{2\tau}[(\beta_{,2})^{2} + (\gamma_{,2})^{2} - \beta_{,23} - \gamma_{,24}] = 0 \qquad (4.23)$$

$$R_{22} \equiv 0 \qquad (4.24)$$

$$R_{23} = -\tfrac{1}{2}\, \beta_{,22} = 0 \;, \qquad (4.25)$$

$$R_{24} = -\tfrac{1}{2}\, \gamma_{,22} = 0 \;, \qquad (4.26)$$

$$R_{33} = -\nabla^{2}\tau - (\beta_{,3} + \beta\,\tau_{,3} - \gamma\,\tau_{,4})_{,2} + \tfrac{1}{2}\,(\beta_{,2})^{2} = 0 \;, \qquad (4.27)$$

$$R_{34} = -[\tfrac{1}{2}(\gamma_{,3} + \beta_{,4}) + \beta\,\tau_{,4} + \gamma\,\tau_{,3}]_{,2} + \tfrac{1}{2}\,\beta_{,2}\,\gamma_{,2} = 0 \ , \tag{4.28}$$

$$R_{44} = -\nabla^2\tau - (\gamma_{,4} - \beta\,\tau_{,3} + \gamma\,\tau_{,4})_{,2} + \tfrac{1}{2}(\gamma_{,2})^2 = 0 \ , \tag{4.29}$$

where

$$\nabla^2\tau \equiv \tau_{,33} + \tau_{,44} \ . \tag{4.30}$$

From (4.25) and (4.26) it follows that

$$\beta = x^2\,\xi + \xi_0 \ , \qquad \gamma = x^2\,\eta + \eta_0 \ , \tag{4.31}$$

where the ξ's and η's are functions of x^1, x^3 and x^4. From equations (4.27) - (4.29) one obtains:

$$-\nabla^2\tau - \xi_{,3} - \xi\,\tau_{,3} + \eta\,\tau_{,4} + \tfrac{1}{2}\,\xi^2 = 0 \ , \tag{4.32}$$

$$-\nabla^2\tau - \eta_{,4} + \xi\,\tau_{,3} - \eta\,\tau_{,4} + \tfrac{1}{2}\,\eta^2 = 0 \ , \tag{4.33}$$

$$-\tfrac{1}{2}(\xi_{,4} + \eta_{,3}) - \xi\,\tau_{,4} - \eta\,\tau_{,3} + \tfrac{1}{2}\,\xi\,\eta = 0 \ . \tag{4.34}$$

If $\xi^2 + \eta^2 \neq 0$ these equations are equivalent to

$$\tau_{,3} = \tfrac{1}{4}\,\xi - \tfrac{1}{4}[\log(\xi^2+\eta^2)]_{,3} + \tfrac{1}{2}[\mathrm{Arctan}(\eta/\xi)]_{,4} \ , \tag{4.35}$$

$$\tau_{,4} = \tfrac{1}{4}\,\eta - \tfrac{1}{4}[\log(\xi^2+\eta^2)]_{,4} - \tfrac{1}{2}[\mathrm{Arctan}(\eta/\xi)]_{,3} \ , \tag{4.36}$$

$$\nabla^2\tau = \tfrac{1}{4}(\xi^2 + \eta^2) - \tfrac{1}{2}(\xi_{,3} + \eta_{,4}) \ . \tag{4.37}$$

The compatibility equations for (4.35) and (4.36) are

$$\nabla^2 \operatorname{Arctan}(\eta/\xi) = \tfrac{1}{2}(\eta_{,3} - \xi_{,4}) \ , \tag{4.38}$$

$$\nabla^2 \log(\xi^2 + \eta^2) = 3(\xi_{,3} + \eta_{,4}) - (\xi^2 + \eta^2) \ . \tag{4.39}$$

To solve the above set of equations it is convenient to put

$$\xi = e^{-2\tau} u \ , \qquad \eta = e^{-2\tau} v \ . \tag{4.40}$$

In terms of these variables, equations (4.32), (4.33), (4.34), (4.38), (4.39) become:

$$u_{,3} - v_{,4} = \tfrac{1}{2}(u^2 - v^2)\, e^{-2\tau} \ , \tag{4.41}$$

$$u_{,4} + v_{,3} = uve^{-2\tau} \ , \tag{4.42}$$

$$\nabla^2\tau = \tfrac{1}{4} e^{-4\tau}(u^2+v^2) - \tfrac{1}{2}(\xi_{,3} + \eta_{,4}) \ , \tag{4.43}$$

$$\nabla^2 \operatorname{Arctan}(v/u) = \tfrac{1}{2}(\eta_{,3} - \xi_{,4}) \ , \tag{4.44}$$

$$\nabla^2 \log(u^2 + v^2) = \xi_{,3} + \eta_{,4} \ . \tag{4.45}$$

If τ is eliminated by combining (4.41) and (4.42), the result is

$$(u^2-v^2)(u_{,4}+v_{,3}) - 2uv(u_{,3}-v_{,4}) = 0 \ , \tag{4.46}$$

and this is equivalent to

$$\left(\frac{v}{u^2+v^2}\right)_{,3} - \left(\frac{u}{u^2+v^2}\right)_{,4} = 0 \qquad (4.47)$$

provided $u^2 + v^2 \neq 0$.

If it is assumed that $u^2 + v^2 \neq 0$, then there exists a $\varphi(x^1, x^3, x^4)$ such that

$$\frac{u}{u^2+v^2} = -\varphi_{,3} \ , \qquad \frac{v}{u^2+v^2} = -\varphi_{,4} \ , \qquad (4.48)$$

and therefore

$$u = \frac{-\varphi_{,3}}{\varphi_{,3}^2 + \varphi_{,4}^2} \ , \qquad v = \frac{-\varphi_{,4}}{\varphi_{,3}^2 + \varphi_{,4}^2} \ , \qquad (4.49)$$

$$e^{-2\tau} = 2\nabla^2\varphi \ , \qquad (4.50)$$

$$\nabla^2\tau = \frac{e^{-4\tau}}{4(\varphi_{,3}^2 + \varphi_{,4}^2)} - \tfrac{1}{2}(\xi_{,3} + \eta_{,4}) \ , \qquad (4.51)$$

$$\nabla^2 \operatorname{Arctan}(\varphi_{,4}/\varphi_{,3}) = \tfrac{1}{2}(\eta_{,3} - \xi_{,4}) \ , \qquad (4.52)$$

$$-\nabla^2 \log(\varphi_{,3}^2 + \varphi_{,4}^2) = \xi_{,3} + \eta_{,4} \ , \qquad (4.53)$$

where

$$\xi = \frac{2\varphi_{,3}\nabla^2\varphi}{\varphi_{,3}^2 + \varphi_{,4}^2} \ , \qquad \eta = \frac{-2\varphi_{,4}\nabla^2\varphi}{\varphi_{,3}^2 + \varphi_{,4}^2} \ . \qquad (4.54)$$

It can be easily verified that (4.51) and (4.52) are identically satisfied. However, equations (4.49), (4.50) and (4.52) imply that φ must be a solution of

$$\nabla^2 \log[\nabla^2\varphi(\varphi_{,3}^2 + \varphi_{,4}^2)] + \frac{2(\nabla^2\varphi)^2}{\varphi_{,3}^2 + \varphi_{,4}^2} = 0 \quad . \tag{4.55}$$

For every solution of (4.55) it is true that a determination of ξ , η and τ is given by (4.50) and (4.54). Equation (4.55) is a fourth order partial differential equation and its general solution contains four arbitrary functions. The dependence on two of these is easily obtained. If we let

$$z = x^3 + ix^4 \quad , \qquad \bar{z} = x^3 - ix^4 \quad , \tag{4.56}$$

then (4.55) becomes

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log\left[\frac{\partial^2\varphi}{\partial z \partial \bar{z}} \left(\frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial\varphi}{\partial \bar{z}}\right)\right] + \frac{2\left(\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)^2}{\frac{\partial\varphi}{\partial z}\;\frac{\partial\varphi}{\partial\bar{z}}} = 0 \quad , \tag{4.57}$$

If $\mu = \mu(z)$, $\nu = \nu(\bar{z})$ are arbitrary functions, then

$$\left.\begin{aligned} &\frac{\partial^2}{\partial z\partial\bar{z}} \log\left(\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\;\frac{\partial\varphi}{\partial z}\;\frac{\partial\varphi}{\partial\bar{z}}\right) + \frac{2\left(\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)^2}{\frac{\partial\varphi}{\partial z}\;\frac{\partial\varphi}{\partial\bar{z}}} = \\ &\frac{d\mu}{dz}\,\frac{d\nu}{d\bar{z}}\left[\frac{\partial^2}{\partial\mu\partial\nu}\log\left(\frac{\partial^2\varphi}{\partial\mu\partial\nu}\;\frac{\partial\varphi}{\partial\mu}\;\frac{\partial\varphi}{\partial\nu}\right) + \frac{2\left(\frac{\partial^2\varphi}{\partial\nu\partial\mu}\right)}{\frac{\partial\varphi}{\partial\mu}\;\frac{\partial\varphi}{\partial\nu}}\right] \quad . \end{aligned}\right\} \tag{4.58}$$

If $\varphi = \varphi(\mu,\nu)$ is any solution of

$$\frac{\partial^2}{\partial\mu\partial\nu}\log\left(\frac{\partial^2\varphi}{\partial\mu\partial\nu}\;\frac{\partial\varphi}{\partial\mu}\;\frac{\partial\varphi}{\partial\nu}\right) + \frac{2\left(\frac{\partial^2\varphi}{\partial\nu\partial\mu}\right)}{\frac{\partial\varphi}{\partial\mu}\;\frac{\partial\varphi}{\partial\nu}} = 0 \quad , \tag{4.59}$$

then

$$\varphi = \varphi(\mu(z), \nu(\bar{z})) \tag{4.60}$$

will also be a solution of (4.57).

It seems unlikely that the general solution to (4.55) can be presented in any reasonably simple form. For example, if it is assumed that $\varphi = \varphi(x^3)$ then (4.54) becomes:

$$[\log \varphi_{,33}(\varphi_{,3})^2]_{,33} + 2\left(\frac{\varphi_{,33}}{\varphi_{,3}}\right)^2 = 0 . \tag{4.61}$$

The substitution

$$V = \frac{\varphi_{,3}}{\varphi_{,33}} , \tag{4.62}$$

reduces (4.60) to

$$V_{,33} - \frac{1}{V}[(V_{,3})^2 - 3V_{,3} + 2] = 0 . \tag{4.63}$$

Equation (4.63) is equivalent to

$$\frac{p\ d\ p}{(p-1)(p-2)} = \frac{dV}{V} , \tag{4.64}$$

where $p = V_{,3}$. This equation is easily integrated and the result is

$$\frac{(p-2)^2}{(p-1)} = c\ V , \tag{4.65}$$

where c is an arbitrary constant. It is possible to integrate (4.65) and obtain x^3 as an explicit function of V. Unfortunately, the expression is too complicated to be inverted so it is not possible to

obtain a general expression for $\varphi_{,3}$. However, it is clear from (4.63) that two particular solutions are

$$V = x^3 + c \quad , \qquad V = 2x^3 + c \quad , \tag{4.66}$$

and from (4.50), (4.54) and (4.62) it is seen that these solutions lead to the following possibilities:

(i) $$\varphi_{,3} = c_1 x^3 + c_2 \quad ,$$

$$\tau = -\tfrac{1}{2}\log(2c_1) \quad , \tag{4.67}$$

$$\xi = \frac{-2c_1}{c_1 x^3 + c_2} \quad , \qquad \eta = 0 \quad ;$$

(ii) $$\varphi_{,3} = c_1 \sqrt{2x^3} + c_2$$

$$\tau = -\tfrac{1}{2}\log\left(\frac{2c_1}{\sqrt{2x^3} + c_2}\right) \tag{4.68}$$

$$\xi = \frac{-2}{2x^3 + c_1} \quad , \qquad \eta = 0 \quad ;$$

where c_1 is any positive constant and c_2 is arbitrary.

Returning to the general case, we shall now consider the equations

$$R_{13} = 0 \quad , \qquad R_{14} = 0 \quad , \qquad R_{11} = 2e^{2\tau} \quad . \tag{4.69}$$

It can be shown (see appendix) that

$$R_{11} = (x^2)^2 R_{11}^{(2)} + x^2 R_{11}^{(1)} + R_{11}^{(0)} , \qquad (4.70)$$

$$R_{13} = x^2 R_{13}^{(1)} + R_{13}^{(0)} , \qquad (4.71)$$

$$R_{14} = x^2 R_{14}^{(1)} + R_{14}^{(0)} , \qquad (4.72)$$

where the new quantities are independent of x^2. Hence, we have the following group of equations:

$$R_{11}^{(2)} = 0 , \quad R_{11}^{(1)} = 0 , \quad R_{11}^{(0)} = 2e^{2\tau} , \qquad (4.73)$$

$$R_{13}^{(1)} = 0 , \quad R_{13}^{(0)} = 0 , \qquad (4.74)$$

$$R_{14}^{(1)} = 0 , \quad R_{14}^{(0)} = 0 . \qquad (4.75)$$

The quantities $R_{11}^{(2)}$, $R_{13}^{(1)}$, $R_{14}^{(1)}$ involve only the parameters ξ , η and τ. A lengthy calculation shows that every solution of (4.32) - (4.34) automatically insures that the quantities $R_{11}^{(2)}$, $R_{13}^{(1)}$ and $R_{14}^{(1)}$ are identically zero. The computations which lead to the explicit expression for R_{11} are extremely lengthy and a completely general solution for this case seems difficult to obtain. However, it is possible to obtain the general solution for the following two sub-cases:

<u>Case (i)</u> $\qquad \alpha_{,22} = 0$.

The explicit expression for the equation $R_{11}^{(2)} = 0$ is

$$\begin{aligned} &\nabla^2 \alpha_{,22} - 3(\xi \alpha_{,223} + \eta \alpha_{,224}) + \\ &+ \alpha_{,22}[2(\xi^2+\eta^2)-(\xi_{,3}+\eta_{,4})] - e^{2\tau}(\xi_{,4}-\eta_{,3})^2 = 0 , \end{aligned} \qquad (4.76)$$

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where

$$\alpha_{,22} = \tfrac{1}{2} e^{2\tau} (\xi^2 + \eta^2 - \xi_{,3} - \eta_{,4}) \quad . \tag{4.23a}$$

Since $\alpha_{,22}$ is zero, there exists a function $u = u(x^1, x^3, x^4)$ such that

$$\xi = u_{,3} \quad , \qquad \eta = u_{,4} \quad , \tag{4.77}$$

and

$$\nabla^2 u = (u_{,3})^2 + (u_{,4})^2 \quad . \tag{4.78}$$

Equation (4.78) implies that e^{-u} is a harmonic function. Hence, the transformation

$$\bar{x}^3 = e^{-u} \quad , \qquad \bar{x}^4 = \text{harm. conj. } (e^{-u}) \quad , \tag{4.79}$$

preserves the form of g_{ij}, but in the new reference frame we will have,

$$\xi = -\frac{1}{x^3} \quad , \qquad \eta = 0 \quad . \tag{4.80}$$

If (4.80) is applied to (4.35), (4.36) the result is

$$\tau = \tfrac{1}{4} \log x^3 + \tau_0(x^1) \quad . \tag{4.81}$$

It can be shown that there is no loss of generality in putting τ_0 and η_0 equal to zero. Hence, we have

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$$\alpha = \alpha_{,2}\, x^2 + \alpha_0 \;, \qquad \beta = \frac{-x^2}{x^3} + \xi_0 \;,$$

$$\tau = \tfrac{1}{4}\log x^3 \;, \qquad \gamma = 0 \;, \tag{4.82}$$

where $\alpha_{,2}$ and α_0 are independent of x^2. Equations (4.73) and (4.74) imply

$$\alpha_{,23} = \tfrac{1}{2}\sqrt{x^3}\;\xi_{0,44} \;, \tag{4.83}$$

$$\alpha_{,24} = -\tfrac{1}{2}\sqrt{x^3}\left(\frac{3}{2x^3}\,\xi_{0,4} + \xi_{0,43}\right) . \tag{4.84}$$

The integration of (4.84) yields

$$\alpha_{,2} = -\tfrac{1}{2}\sqrt{x^3}\left(\frac{3}{2x^3}\,\xi_0 + \xi_{0,3}\right) + f(x^1, x^3) \quad . \tag{4.85}$$

But ξ_0 can be replaced by ξ_0 plus any function of x^1 and x^3 by transformations of the form

$$x^2 = \bar{x}^2 + h(x^1, x^3) \quad . \tag{4.86}$$

Hence, there is no loss of generality in putting $f(x^1, x^3)$ equal to zero. Equation (4.85) may be written

$$\alpha_{,2} = \frac{-1}{2x^3}\,\left[(x^3)^{3/2}\,\xi_0\right]_{,3} \quad . \tag{4.87}$$

If the above expression for $\alpha_{,2}$ is substituted into (4.83), the result is

$$\nabla^2 u - \frac{u_{,3}}{x^3} = 0 \;, \tag{4.88}$$

where

$$u \equiv \xi_0 \, (x^3)^{3/2} \quad . \tag{4.89}$$

The general solution to (4.88) is

$$u = x^3 \int_0^\pi [F(x^3 \cos\varphi + ix^4) + G(x^3\cos\varphi - ix^4)] \cos\varphi \, d\varphi \quad , \tag{4.90}$$

where F and G are arbitrary functions of their arguments. Hence,

$$\xi_0 = \frac{1}{\sqrt{x^3}} \int_0^\pi (F + G) \cos\varphi \, d\varphi \quad , \tag{4.91}$$

$$\alpha_{,2} = -\tfrac{1}{2} \int_0^\pi (F' + G') \, d\varphi \quad . \tag{4.92}$$

The first equation of set (4.73) has been satisfied and the second one reduces to an identity. However, the third equation is

$$\begin{aligned} \nabla^2 \alpha_0 - \frac{\alpha_{0,3}}{x^3} + \frac{\alpha_0}{(x^3)^2} &= 2 + \xi_{0,3}\, \alpha_{,2} \\ &+ 2\, \xi_0\, \alpha_{,23} + \xi_{0,13} + \frac{\sqrt{x^3}}{2} (\xi_{0,4})^2 \quad . \end{aligned} \tag{4.93}$$

Since ξ_0 and $\alpha_{,2}$ are known, the general solution of (4.93) is

$$\begin{aligned} \alpha_0 = x^3 \int_0^\pi &[F^*(x^3\cos\varphi + ix^4) + G^*(x^3\cos\varphi - ix^4)] \, d\varphi \\ &+ \bar{\alpha}_0 \quad , \end{aligned} \tag{4.94}$$

where F* and G* are arbitrary, and $\bar{\alpha}_0$ is a particular integral of (4.93).

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It is interesting to note that the field equations place no restrictions on the variable x^1.

Examples:

(i) $F = G = \text{constant}$, implies

$$\left.\begin{aligned} &\xi_0 = 0 \ , \qquad \alpha_{,2} = 0 \ , \\ &\alpha_0 = x^3 \int_0^\pi (F^* + G^*)\, d\varphi + 2(x^3)^2 \ . \end{aligned}\right\} \quad (4.95)$$

Hence,

$$g_{ij} = \begin{pmatrix} \alpha_0 & 1 & \frac{-x^2}{x^3} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-x^2}{x^3} & 0 & \frac{1}{\sqrt{x^3}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{x^3}} \end{pmatrix} \ . \quad (4.96)$$

(ii) $F(\sigma) = G(\sigma) = a(x^1)\sigma$, implies

$$\xi_0 = \pi\, a \sqrt{x^3} \ , \qquad \alpha_{,2} = -\pi\, a \ ,$$

$$\alpha_0 = x^3 \int_0^\pi (F^* + G^*)\, d\varphi + 2(x^2)^2 \quad (4.97)$$

$$+ 2\pi(a_{,1} + \pi\, a^2)(x^3)^{3/2} \ ,$$

and

$$g_{ij} = \begin{pmatrix} \alpha_0 - \pi a x^2 & 1 & \dfrac{-x^2}{x^3} + \pi a \sqrt{x^3} & 0 \\ 1 & 0 & 0 & 0 \\ \dfrac{-x^2}{x^3} + \pi a \sqrt{x^3} & 0 & \dfrac{1}{\sqrt{x^3}} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{x^3}} \end{pmatrix} . \quad (4.98)$$

<u>Case (ii)</u> $\alpha_{,22} \neq 0 \quad , \quad \nabla^2 \tau = 0 \quad .$

This example shows that $\alpha_{,22}$ need not be zero. Since τ is assumed to be harmonic, a coordinate system exists in which

$$g_{ij} = \begin{pmatrix} 2\alpha & 1 & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix} , \quad (4.16a)$$

and

$$w_i = (e^{\tau}, 0, 0, 0) \quad . \quad (4.8a)$$

As before, the dependence of α, β and γ on x^2 is as follows:

$$\left. \begin{aligned} \alpha &= \tfrac{1}{2} \alpha_{,22} (x^2)^2 + \alpha_2 x^2 + \alpha_0 \quad , \\ \beta &= x^2 \xi + \xi_0 \quad , \quad \gamma = x^2 \eta + \eta_0 \quad , \end{aligned} \right\} \quad (4.99)$$

Equations (4.32) - (4.34) become

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$$\left.\begin{aligned} &\xi_{,3} = \tfrac{1}{2}\,\xi^2 \quad , \qquad \eta_{,4} = \tfrac{1}{2}\,\eta^2 \quad , \\ &\xi_{,4} + \eta_{,3} = \xi\,\eta \ . \end{aligned}\right\} \qquad (4.100)$$

Assuming $\xi\,\eta \neq 0$, equations (4.100) imply

$$\xi = \frac{2}{\varphi(x^4)-x^3} \quad , \qquad \eta = \frac{2}{\psi(x^3)-x^4} \quad , \qquad (4.101)$$

where φ and ψ are solutions of

$$\psi_{,3}(\varphi-x^3)^2 + \varphi_{,4}(\psi-x^4)^2 + 2(\psi-x^4)(\varphi-x^3) \ = \ 0 \quad . \qquad (4.102)$$

For an arbitrary x^3 , let $x^4 = \psi(x^3)$. Equation (4.102) then implies

$$\psi_{,3}(\varphi - x^3)^2 \ = \ 0 \quad . \qquad (4.103)$$

Since ξ and η are assumed to be non-zero, it follows from (4.101) that $\psi_{,3}$ and $\varphi_{,4}$ are non-zero. Hence, we conclude

$$\psi(\varphi(x^4)) = x^4 \quad . \qquad (4.104)$$

By solving these equations when $x^3 = 0$ and $x^4 = 0$, it is possible to prove that the only solutions, (in the general case) are

$$\psi \ = \ ax^3 + b \quad , \qquad \varphi \ = \ \frac{1}{a}\,(x^4 - b) \quad , \qquad (4.105)$$

where a and b are arbitrary functions of x^1. Since translations of the form

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$$\bar{x}^3 = x^3 + f(x^1) \quad , \qquad \bar{x}^4 = x^4 + g(x^1) \quad , \tag{4.106}$$

affect ξ_0, η_0 but not ξ and η , there is no loss of generality by assuming

$$\xi = \frac{2a}{x^4 - ax^3} \quad , \qquad \eta = \frac{2}{ax^3 - x^4} \quad , \tag{4.107}$$

i.e., we may put $b = 0$. Since

$$\xi \, dx^3 + \eta \, dx^4 = -2 \left(\frac{a dx^3 - dx^4}{ax^3 - x^4} \right) \quad , \tag{4.108}$$

the rotation

$$\bar{x}^3 = \frac{ax^3 - x^4}{\sqrt{a^2+1}} \quad , \qquad \bar{x}^4 = \frac{x^3 + ax^4}{\sqrt{a^2+1}} \quad , \tag{4.109}$$

implies

$$x^2(\xi \, dx^3 + \eta \, dx^4) = \frac{-2x^2}{\bar{x}^3} \, d\bar{x}^3 + f(x^1, \bar{x}^3, \bar{x}^4) \, dx^1 \quad . \tag{4.110}$$

Hence, there exists a coordinate system in which

$$\xi = \frac{-2}{x^3} \quad , \qquad \eta = 0 \quad . \tag{4.111}$$

As in the previous case, it is possible to make a transformation which reduces η_0 to zero. The net results are

$$\beta = \frac{-2x^2}{x^3} + \xi_0 \quad , \qquad \gamma = 0 \quad , \tag{4.112}$$

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$$\alpha_{,22} = \frac{1}{(x^3)^2} , \qquad (4.113)$$

the last equation being a direct consequence of (4.111) and (4.23).

The equations $R_{13} = R_{14} = 0$ imply

$$\alpha_{2,3} = \frac{1}{2}\left(\frac{2\xi_0}{(x^3)^2} + \xi_{0,44}\right) , \qquad (4.114)$$

$$\alpha_{2,4} = -\frac{1}{2}\left(\frac{2\xi_{0,4}}{x^3} + \xi_{0,34}\right) . \qquad (4.115)$$

The solution of (4.115) may be taken as

$$\alpha_2 = -\frac{1}{2}\left(\frac{2\xi_0}{x^3} + \xi_{0,3}\right) , \qquad (4.116)$$

(again an arbitrary function is absorbed by an appropriate transformation). If this expression for α_2 is substituted in (4.114), the result is

$$\nabla^2 \xi_0 + \frac{2\xi_{0,3}}{x^3} = 0 . \qquad (4.117)$$

The solution to (4.117) is

$$\xi_0 = \frac{H}{x^3} , \qquad (4.118)$$

where H is an arbitrary harmonic function. Once H is specified, α_2 is determined by (4.116) which reduces to

$$\alpha_2 = \frac{-1}{2(x^3)^2} (x^3 H)_{,3} . \qquad (4.119)$$

The last equation to be considered is $R_{11} = 2e^{2\tau}$. This leads to the following:

$$\left.\begin{aligned}\nabla^2 \alpha_0 - \frac{2\alpha_{0,3}}{x^3} + \frac{2\alpha_0}{(x^3)^2} &= 2e^{2\tau} + 2\,\xi_0\,\alpha_{2,3} \\ &\quad + \alpha_2\,\xi_{0,3} + \tfrac{1}{2}(\xi_{0,4})^2 + \xi_{0,13}\ ,\end{aligned}\right\} \qquad (4.120)$$

where τ is an arbitrary harmonic function. If we let

$$u \equiv \frac{\alpha_0}{x^3}\ , \qquad (4.121)$$

$$F(x^1,x^3,x^4) \equiv \frac{1}{x^3}\left(2e^{2\tau}+2\xi_0\,\alpha_{2,3}+\alpha_2\,\xi_{0,3}+\tfrac{1}{2}(\xi_{0,4})^2+\xi_{0,13}\right)\ , \qquad (4.122)$$

then (4.120) may be written as

$$\nabla^2 u = F(x^1,\ x^3,\ x^4)\ . \qquad (4.123)$$

The general solution of (4.123) is

$$u = \iint F\left(x^1,\ \frac{z+\bar{z}}{2}\ ,\ \frac{z-\bar{z}}{2i}\right) dz\,d\bar{z} + \psi(z) + \varphi(\bar{z})\ , \qquad (4.124)$$

where

$$z = x^3 + ix^4\ , \qquad \bar{z} = x^3 - ix^4\ , \qquad (4.125)$$

and the functions φ , ψ are arbitrary.

A resumé of the above results is as follows: Let H and τ be arbitrary harmonic functions, ξ_0 is determined by (4.118), α_2 by (4.119) and F by (4.122). The function u is determined by (4.124) and α_0 is given by (4.121). Hence, the general solution involves a quadrature and three arbitrary harmonic functions.

Example: Let $H = a\, x^3 x^4$ and $\tau = 0$ where $a = a(x^1)$, then

$$\xi_0 = ax^4 , \qquad \alpha_2 = \frac{-ax^4}{x^3} , \tag{4.126}$$

$$F = \frac{1}{x^3}\left[2 - 2\left(\frac{ax^4}{x^3}\right)^2 + \frac{1}{2} a^2 \right] , \tag{4.127}$$

$$u = \frac{1}{2}\iint \left[\frac{2 + \frac{1}{2} a^2}{z + \bar{z}} + a^2 \frac{(z - \bar{z})^2}{(z + \bar{z})^3} \right] dz d\bar{z} , \tag{4.128}$$

$$\alpha_0 = x^3 u . \tag{4.129}$$

Since ξ and $\alpha_{,22}$ are given by (4.111) and (4.113) respectively, the final result is

$$g_{ij} = \begin{pmatrix} \frac{1}{2}\left(\frac{x^2}{x^3}\right)^2 - ax^4x^2 + \alpha_0 & 1 & \frac{-2x^2}{x^3} + ax^4 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-2x^2}{x^3} + ax^4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.130}$$

CHAPTER V

Discussion of Case III

If $w^i{}_{;i}$ is non-zero, then the equations of integrability become rather unwieldy and we have not been able to attain a general solution for this case. However, it is possible to obtain a non-trivial solution if one assumes that the following equations are valid at P :

$$w_{i;j} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 & \alpha_5 \\ 0 & 0 & -\alpha_5 & -\alpha_2 \end{pmatrix} , \tag{5.1}$$

$$\begin{aligned} w_i &= (1, 0, 0, 0) , \quad a_i = (0, 1, 0, 0) , \\ e_i &= (0, 0, 1, 0) , \quad h_i = (0, 0, 0, 1) . \end{aligned} \tag{5.2}$$

Equations (2.2a), (2.3a) and (2.16) imply

$$a_{i;j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_2 & 0 & 0 \\ -\beta_1 & -\beta_2 & -\beta_3 & -\beta_4 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \end{pmatrix} , \tag{5.3}$$

$$e_{i;j} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & 0 & \alpha_2 & -\alpha_5 \\ 0 & 0 & 0 & 0 \\ \beta_5 & -\alpha_5 & 0 & 0 \end{pmatrix} , \tag{5.4}$$

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$$h_{i;j} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 0 & \alpha_5 & \alpha_2 \\ -\beta_5 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{5.5}$$

The $R_{ijk\ell}$ can be obtained from the integrability equations (3.19) - (3.22), (see appendix) and subjected to the field equation

$$R_{2ij1} + R_{1ij2} + R_{3ij3} + R_{4ij4} = 2\delta'_i\delta'_j \quad . \tag{5.6}$$

Putting (i,j) successively equal to $(3,3)$, $(3,4)$ and $(4,4)$ yields

$$\begin{aligned} 2\,R_{2331} + R_{4334} &= 0 \quad , \\ R_{2341} + R_{1342} &= 0 \quad , \\ 2\,R_{2441} + R_{3443} &= 0 \quad . \end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned} R_{2331} &= R_{2441} \quad , \\ R_{2341} &= R_{2413} \quad . \end{aligned} \tag{5.8}$$

The explicit expression for (5.8) is

$$\begin{aligned} \alpha_2(\beta_3 - \gamma_4) - \alpha_5(\gamma_3 + \beta_4) &= 0 \quad , \\ \alpha_5(\beta_3 - \gamma_4) + \alpha_2(\gamma_3 + \beta_4) &= 0 \quad , \end{aligned} \tag{5.8a}$$

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and since α_2 is not zero it follows that

$$\beta_3 = \gamma_4 \quad , \qquad \gamma_3 = -\beta_4 \quad . \tag{5.9}$$

Also, the expressions

$$\left.\begin{aligned} R_{2134} + R_{2341} + R_{2413} &= 0 \ , \\ R_{2324} - R_{2423} &= 0 \ , \\ R_{1242} + R_{3243} &= 0 \ , \\ R_{1232} + R_{4234} &= 0 \ , \end{aligned}\right\} \tag{5.10}$$

$$\left.\begin{aligned} 2\, R_{2331} + R_{4334} &= 0 \ , \\ R_{3223} + R_{4224} &= 0 \ , \\ R_{2123} - R_{2321} &= 0 \ , \\ R_{2124} - R_{2421} &= 0 \ , \end{aligned}\right\} \tag{5.11}$$

can be easily verified by means of (5.6). Relations (5.10) and (5.11) are equivalent to:

$$\left.\begin{aligned} \alpha_{5,1} &= 2\, \alpha_5\, \beta_3 + \alpha_2\, \beta_4 \ , \\ \alpha_{5,2} &= 3\, \alpha_2\, \alpha_5 \ , \\ \alpha_{5,3} &= 2\, \alpha_2\, \gamma_2 - \alpha_5\, \beta_2 \ , \\ \alpha_{5,4} &= -2\, \alpha_2\, \beta_2 - \alpha_5\, \gamma_2 \ , \end{aligned}\right\} \tag{5.10a}$$

$$\left.\begin{aligned} \alpha_{2,1} &= \alpha_2(\alpha_1 + 2\, \beta_3) - \alpha_5(\beta_4 + \beta_5) \ , \\ \alpha_{2,2} &= 2\, \alpha_2^{\,2} - \alpha_5^{\,2} \\ \alpha_{2,3} &= -\alpha_2\, \beta_2 - 2\alpha_5\, \gamma_2 \ , \\ \alpha_{2,4} &= -\alpha_2\, \gamma_2 + 2\, \alpha_5\, \beta_2 \ . \end{aligned}\right\} \tag{5.11a}$$

From the covariant forms of (5.10a) and (5.11a) we may obtain expressions for $\alpha_{5;ij}$ and $\alpha_{2;ij}$. The compatibility conditions

$$\left.\begin{aligned} \alpha_{5;23} - \alpha_{5;32} &= 0 \,, \\ \alpha_{5;24} - \alpha_{5;42} &= 0 \,, \\ \alpha_{2;23} - \alpha_{2;32} &= 0 \,, \\ \alpha_{2;24} - \alpha_{2;42} &= 0 \,, \end{aligned}\right\} \qquad (5.12)$$

imply:

$$\left.\begin{aligned} \alpha_5\,\beta_{2,2} - 2\,\alpha_2\,\gamma_{2,2} &= \alpha_2\,\alpha_5\,\beta_2 + 4\,\gamma_2(\alpha_5{}^2 - \alpha_2{}^2) \,, \\ 2\,\alpha_2\,\beta_{2,2} + \alpha_5\,\gamma_{2,2} &= \alpha_2\,\alpha_5\,\gamma_2 + 4\,\beta_2(\alpha_2{}^2 - \alpha_5{}^2) \,, \\ \alpha_2\,\beta_{2,2} + 2\,\alpha_5\,\gamma_{2,2} &= 8\,\alpha_2\,\alpha_5\,\gamma_2 + \beta_2\,\alpha_2{}^2 \,, \\ 2\,\alpha_5\,\beta_{2,2} - \alpha_2\,\gamma_{2,2} &= 8\,\alpha_2\,\alpha_5\,\beta_2 - \gamma_2\,\alpha_2{}^2 \,. \end{aligned}\right\} \qquad (5.12a)$$

By straightforward algebra it follows that the only solution to system (5.12a) is

$$\beta_2 = 0 \;, \quad \gamma_2 = 0 \;. \qquad (5.12b)$$

Equations

$$\begin{aligned} \alpha_{5;34} - \alpha_{5;43} &= 0 \\ \alpha_{2;34} - \alpha_{2;43} &= 0 \end{aligned} \qquad (5.13)$$

imply

$$\begin{aligned} \alpha_5(\alpha_5\,\beta_3 - \alpha_2\,\beta_4) &= 0 \\ \alpha_5(\alpha_1\,\alpha_2 - \alpha_5\,\beta_5) &= 0 \end{aligned} \qquad (5.13a)$$

If we now assume that α_5 is non-zero then the identities

$$\left.\begin{aligned} \alpha_{5;13} - \alpha_{5;31} &= 0 , \\ \alpha_{5;14} - \alpha_{5;41} &= 0 , \\ \alpha_{2;13} - \alpha_{2;31} &= 0 , \\ \alpha_{2;14} - \alpha_{2;41} &= 0 , \end{aligned}\right\} \quad (5.14)$$

coupled with (5.13a) imply

$$\left.\begin{aligned} \beta_{3,3} &= - \beta_1 \alpha_2 , \\ \beta_{4,3} &= - \beta_1 \alpha_5 , \\ \beta_{3,4} &= - \gamma_1 \alpha_2 , \\ \beta_{4,4} &= - \gamma_1 \alpha_5 . \end{aligned}\right\} \quad (5.15)$$

Since we are assuming that α_5 is non-zero, the last equation of (5.13a) implies

$$\begin{aligned} \alpha_2 \alpha_{1,3} - \alpha_5 \alpha_{5,3} &= 0 , \\ \alpha_2 \alpha_{1,4} - \alpha_5 \beta_{5,4} &= 0 . \end{aligned} \quad (5.16)$$

However, the relations

$$\left.\begin{aligned} R_{2131} + R_{4134} &= 0 , \\ R_{2141} + R_{3143} &= 0 , \\ R_{3134} - R_{3431} &= 0 , \\ R_{4134} - R_{3441} &= 0 , \end{aligned}\right\} \quad (5.17)$$

give us the following set of conditions:

$$\left.\begin{aligned}
\beta_{3,3} + \beta_{4,4} + \alpha_{1,3} &= \gamma_1\,\alpha_5 - 2\,\alpha_2\,\beta_1 \;, \\
\beta_{3,4} - \beta_{4,3} + \alpha_{1,4} &= -\,\beta_1\,\alpha_5 - 2\,\alpha_2\,\gamma_1 \;, \\
\beta_{3,4} - \beta_{4,3} + \beta_{5,3} &= -\,\gamma_1\,\alpha_2 \;, \\
\beta_{3,3} + \beta_{4,4} - \beta_{5,4} &= -\,\beta_1\,\alpha_2 \;.
\end{aligned}\right\} \qquad (5.17a)$$

However, (5.16) together with (5.17a) requires that

$$\beta_1 = \gamma_1 = 0 \;. \qquad (5.18)$$

If the first equation in the set (5.13a) is differentiated with respect to x^1, then $\alpha_5 \neq 0$ implies

$$\alpha_5\,\beta_{3,1} - \alpha_2\,\beta_{4,1} = \beta_4\,\alpha_{2,1} - \beta_3\,\alpha_{5,1} \;. \qquad (5.19)$$

Expressions for $\beta_{3,1}$ and $\beta_{4,1}$ may be obtained from the relations

$$\begin{aligned}
R_{3113} + R_{4114} &= 2 \;, \\
R_{3114} - R_{1431} &= 0 \;,
\end{aligned} \qquad (5.20)$$

the results being:

$$\begin{aligned}
\beta_{3,1} &= \beta_3(\beta_3 - \alpha_1) - \beta_4^{\,2} - 1 \;, \\
\beta_{4,1} &= \beta_4(2\,\beta_3 - \alpha_1) \;.
\end{aligned} \qquad (5.20a)$$

The net result of (5.19), (5.20a), (5.13a), (5.10a) and (5.11a) is

$$\alpha_5 = 0 \ . \tag{5.21}$$

Much of the preceding analysis depended on α_5 being non-zero, however, some of the results were completely general. To avoid confusion we present the following recapitulation.

$$\left.\begin{aligned}
&\alpha_5 = 0 \ , \quad \beta_2 = 0 \ , \quad \gamma_2 = 0 \ , \\
&\beta_4 = 0 \ , \quad \gamma_3 = 0 \ , \quad \beta_3 = \gamma_4 \ , \\
&\alpha_{2,1} = \alpha_2(\alpha_1 + 2\,\beta_3) \ , \\
&\alpha_{2,2} = 2\,\alpha_2^2 \ , \\
&\alpha_{2,3} = 0 \ , \\
&\alpha_{2,4} = 0 \ , \\
&\beta_{3,3} + \alpha_{1,3} = -\,2\,\alpha_2\,\beta_1 \ , \\
&\beta_{3,4} + \alpha_{1,4} = -\,2\,\alpha_2\,\gamma_1 \ , \\
&\beta_{3,4} + \beta_{5,3} = -\,\alpha_2\,\gamma_1 \ , \\
&\beta_{3,3} - \beta_{5,4} = -\,\alpha_2\,\beta_1 \ .
\end{aligned}\right\} \tag{5.22}$$

Equations (5.12a) and (5.13a) are identically satisfied but (5.14) implies

$$\left.\begin{aligned}
&\alpha_{1,3} + 2\,\beta_{3,3} = -\,2\,\beta_1\,\alpha_2 \ , \\
&\alpha_{1,4} + 2\,\beta_{3,4} = -\,2\,\gamma_1\,\alpha_2 \ .
\end{aligned}\right. \tag{5.14a}$$

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The simultaneous solution to (5. 14a) and (5.22) is:

$$\left.\begin{aligned} \beta_{3,3} &= 0 \ , & \beta_{3,4} &= 0 \ , \\ \alpha_{1,3} &= -2\,\beta_1\,\alpha_2 \ , & \alpha_{1,4} &= -2\,\gamma_1\,\alpha_2 \ , \\ \beta_{5,3} &= -\gamma_1\,\alpha_2 \ , & \beta_{5,4} &= \beta_1\,\alpha_2 \ . \end{aligned}\right\} \tag{5. 23}$$

Since

$$\begin{aligned} \alpha_{1;34} - \alpha_{1;43} &= 0 \\ \beta_{5;34} - \beta_{5,43} &= 0 \end{aligned} \tag{5.24}$$

it follows from (5.23) that

$$\begin{aligned} \beta_{1,4} - \gamma_{1,3} &= 0 \\ \beta_{1,3} + \gamma_{1,4} &= 0 \ . \end{aligned} \tag{5.24a}$$

The first equation in set (5.24a) is sufficient to insure that the second equation of (5.20) is identically satisfied, but the first equation of (5.20) together with the second equation of (5.24a) implies

$$\beta_{3,1} = \beta_3(\beta_3 - \alpha_1) - 1 \ . \tag{5.25}$$

However, (5.25) allows us to evaluate

$$\begin{aligned} \beta_{3;13} - \beta_{3;31} &= 0 \ , \\ \beta_{3;14} - \beta_{3;41} &= 0 \ , \end{aligned} \tag{5.26}$$

and the results are:

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$$\beta_1(2\,\alpha_2\,\beta_3 - \beta_{3,2}) = 0 \;,$$
$$\gamma_1(2\,\alpha_2\,\beta_3 - \beta_{3,2}) = 0 \;. \tag{5.26a}$$

But the identity

$$R_{2313} = R_{1323} \;, \tag{5.27}$$

gives

$$\beta_{3,2} = \alpha_2\,\beta_3 \;. \tag{5.27a}$$

Hence, (5.26a) reduces to

$$\beta_1\,\alpha_2\,\beta_3 = 0 \;,$$
$$\gamma_1\,\alpha_2\,\beta_3 = 0 \;. \tag{5.26b}$$

Since α_2 is not zero and (5.25) demands that β_3 be non-zero, then it follows that

$$\beta_1 = \gamma_1 = 0 \;. \tag{5.28}$$

There are only two more relations involving the $R_{ijk\ell}$ which are not identically satisfied, viz.:

$$R_{12} = R_{2121} + R_{3123} + R_{4124} = 0 \;,$$
$$R_{1234} - R_{3412} = 0 \;. \tag{5.29}$$

These equations reduce to

$$\alpha_{1,2} = -\,\alpha_1\,\alpha_2$$
$$\beta_{5,2} = -\,\beta_5\,\alpha_2 \;. \tag{5.29a}$$

The integrability conditions (3.19) - (3.22), the identities (3.24), (3.25) and field equation (3.26) are now expressed in terms of the invariants α_1, α_2, β_3 and β_5. The relevant results are contained in the following set:

$$w_{i;j} = \alpha_1 w_i w_j + \alpha_2 (w_i a_j - e_i e_j - h_i h_j) , \tag{5.30}$$

$$a_{i;j} = - \alpha_1 a_i w_j - \alpha_2 a_i a_j - \beta_3 (e_i e_j + h_i h_j) , \tag{5.31}$$

$$e_{i;j} = \beta_3 w_i e_j + \alpha_2 a_i e_j + \beta_5 h_i w_j , \tag{5.32}$$

$$h_{i;j} = \beta_3 w_i h_j + \alpha_2 a_i h_j - \beta_5 e_i w_j , \tag{5.33}$$

$$\alpha_{1,i} = (\alpha_{1,j} a^j) w_i - \alpha_1 \alpha_2 a_j , \tag{5.34}$$

$$\beta_{5,i} = (\beta_{5,j} a^j) w_i - \beta_5 \alpha_2 a_j , \tag{5.35}$$

$$\alpha_{2,i} = \alpha_2 (\alpha_1 + 2 \beta_3) w_i + 2 \alpha_2^2 a_i , \tag{5.36}$$

$$\beta_{3,i} = [\beta_3 (\beta_3 - \alpha_1) - 1] w_i + \alpha_2 \beta_3 a_i . \tag{5.37}$$

From (5.30) and (5.36) it follows that

$$\alpha_{2,i} w_j - \alpha_{2,j} w_i = - 2 \alpha_2 (w_{i,j} - w_{j,i}) . \tag{5.38}$$

This equation may be easily integrated and the result is:

$$w_i = \sqrt{\alpha_2} w_{,i} , \tag{5.39}$$

where w is some invariant.

If we now choose a coordinate system such that w^i has the form

$$w^i = (0, w^2, 0, 0) \quad , \tag{5.40}$$

then the fact that w^i is a null vector implies that w is independent of x^2. However, for a non-trivial solution w must not be a constant and there is no loss of generality in assuming that w is a function of x^1. Since α_2 is not zero, (5.36) implies that α_2 is a function of x^2. Hence, the coordinate transformation

$$\bar{x}^1 = w \quad , \quad \bar{x}^2 = -\frac{1}{\sqrt{\alpha_2}} \quad , \quad \bar{x}^3 = x^3 \quad , \quad \bar{x}^4 = x^4 \quad , \tag{5.41}$$

is permissible, and in the new reference frame

$$w_i = (-\frac{1}{x^2}, 0, 0, 0) \quad , \tag{5.42}$$

$$a_i = (\frac{x^2}{2}(\alpha_1 + 2\beta_3), -x^2, 0, 0) \quad . \tag{5.43}$$

Furthermore, it is clear that the reference frame may be chosen so that

$$e_i = (\xi, 0, -x^2, 0) \quad , \tag{5.44}$$

$$h_i = (\eta, 0, 0, -x^2) \quad , \tag{5.45}$$

where ξ and η are as yet undetermined.

The invariants α_1, β_3, β_5, ξ and η are subject to

equations (5.30) - (5.37). For example (5.34) implies

$$\alpha_{1,2} = \frac{\alpha_1}{x^2} , \tag{5.46}$$

but it places no restrictions on $\alpha_{1,1}$. Hence, the general solution to this equation is

$$\alpha_1 = 2\,\bar{\alpha}_1\, x^2 , \tag{5.47}$$

where $\bar{\alpha}_1$ is an arbitrary function of x^1.

Similarly, the general solution to (5.35) is

$$\beta_5 = \bar{\beta}_5\, x^2 \tag{5.48}$$

$\bar{\beta}_5$ an arbitrary function of x^1.

The result given by (5.37) is

$$\beta_3 = \bar{\beta}_3 / x^2 , \tag{5.49}$$

$\bar{\beta}_3$ being a function of x^1 satisfying

$$\bar{\beta}_{3,1} = 3\,\bar{\alpha}_1\,\bar{\beta}_3 + 1 . \tag{5.50}$$

Hence

$$\bar{\beta}_3 = e^{3A} \int e^{-3A}\, dx^1 , \tag{5.51}$$

where

$$A \equiv \int \bar{\alpha}_1\, dx^1 . \tag{5.52}$$

Since

$$e_{i,j} - e_{j,i} = \beta_3 \, F_{ij} - \beta_5 \, {*F}_{ij} + \alpha_2 (a_i \, e_j - a_j \, e_i) \tag{5.53}$$

it follows that

$$\begin{aligned} \xi_{,2} &= \xi \,/\, x^2 \\ \xi_{,3} &= -\,\bar{\alpha}_1 \, x^2 \\ \xi_{,4} &= -\,\bar{\beta}_5 \, x^2 \end{aligned} \tag{5.54}$$

and the general solution to this set is

$$\xi = -x^2(\bar{\alpha}_1 \, x^3 + \bar{\beta}_5 \, x^4 + c_1) \quad , \tag{5.55}$$

$c_1 = c_1(x^1)$ being arbitrary.

A similar argument applied to $h_{i,j}$ leads to

$$\eta = -x^2(\bar{\alpha}_1 \, x^4 - \bar{\beta}_5 \, x^3 + c_2) \quad . \tag{5.56}$$

Summary of Results

If space-time is such that there exists a reference frame in which the components of g_{ij} are given by

$$g_{ij} = \begin{pmatrix} 2\alpha & 1 & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ \beta & 0 & (x^2)^2 & 0 \\ \gamma & 0 & 0 & (x^2)^2 \end{pmatrix} \quad , \tag{5.57}$$

where

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$$\alpha = -\bar{\alpha}_1 x^2 - \frac{\bar{\beta}_3}{x^2} + \frac{\beta^2 + \gamma^2}{(x^2)^2} , \tag{5.58}$$

$$\beta = (x^2)^2 (\bar{\alpha}_1 x^3 + \bar{\beta}_5 x^4 + c_1) , \tag{5.59}$$

$$\gamma = (x^2)^2 (\bar{\alpha}_1 x^4 - \bar{\beta}_5 x^3 + c_2) , \tag{5.60}$$

then the electromagnetic field is null and the electromagnetic tensor is given by

$$F_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.61}$$

CHAPTER VI

Conclusion

As mentioned earlier in this thesis, Rainich [6] presented a formulation of relativistic electrodynamics in which the classical electromagnetic field tensor played no role. Unfortunately, the Rainich formulation is valid only for non-null fields and fails to retain meaning for null fields. This fact lead some investigators to believe (see Rainich [6], Witten [8]) that nontrivial null fields might be of no physical interest. Since, Hlavatý [9] has now produced an equivalent formulation which includes the possibility of null fields, there is no reason why one should discard such solutions on an a priori basis.

Although many examples of null fields are known within classical electromagnetic theory, very few are known within the frame work of classical relativistic electrodynamics. As far as we are aware, no significant attempt has ever been made to find all solutions of the field equations that determine null fields. All particular examples of such fields have been obtained by placing stringent a priori conditions on the metric tensor. For example, Peres [14] considered a line element of the form

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &\quad - 2\, f(x,y,z+t)\, (dz + dt)^2 \quad . \end{aligned} \tag{6.1}$$

This situation can be shown to lead to a special solution of the type considered in Case I. Also, Pandya and Vaidya [15] showed that a coordinate system can be chosen in which the metric tensor has the following representation:

$$g_{ij} = \begin{pmatrix} 2\alpha & a & \beta & \gamma \\ a & 0 & 0 & 0 \\ \beta & 0 & e^{-2\tau} & 0 \\ \gamma & 0 & 0 & e^{-2\tau} \end{pmatrix} . \qquad (6.2)$$

They then assumed that

$$a = 1 , \qquad \tau = \tau(x^1,x^3,x^4) ,$$
$$\beta = \beta(x^1,x^3,x^4) , \qquad \gamma = \gamma(x^1,x^3,x^4) , \qquad (6.3)$$

and obtained particular solutions of the type considered in Case I.

At the outset of the preceding work, it was our aim to explicitly determine all solutions of the null field equations. Unfortunately, this objective was not attained. In all but one of the different cases that must be considered, the null vector w_i was the product of a scalar function and a gradient. There is therefore some reason to suppose that w_i must have this form in all cases. If this is true, certain preliminary calculations indicate that it might be possible to determine the general solution for Case III. This would then leave only one sub-case of Case II left unsolved, viz. the case for which $\nabla^2\tau \neq 0$, $\alpha_{,22} \neq 0$. For this case, we have no suggestion of a possible method of attack which might yield the general solution.

Our procedure has produced extensive classes of solutions of the equations that determine null fields. In many of the known solutions, harmonic functions play an important role. In addition to these, our procedure has produced extensive classes of solutions in which other

functions play dominant roles. It is hoped that the physical implications of our solutions will be investigated and that they will provide some insight into the physical nature of null fields.

A possible by-product of our results is due to Bonnor [10]. He has shown that every static solution of

$$\begin{aligned} R_{ij} + \tfrac{1}{2} U_{,i} U_{,j} &= 0 , \\ U^{;i}{}_{;i} &= 0 , \end{aligned} \tag{6.4}$$

leads to a static electromagnetic solution and a static exterior gravitational solution. Since the general solution to (6.4) has been obtained for the case when $U_{,i}$ is a null vector, it is possible that the solutions of Case I might lead to some new gravitational and non-null electromagnetic solutions.

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APPENDIX

Part A

The following set of point equations apply to Cases I and III:

$$\left.\begin{array}{ll} w_i = (1,0,0,0) , & a_i = (0,1,0,0) , \\ e_i = (0,0,1,0) , & h_i = (0,0,0,1) . \end{array}\right\} \quad (A.1)$$

$$w_{i;j} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ -\alpha_3 & 0 & -\alpha_2 & \alpha_5 \\ -\alpha_4 & 0 & -\alpha_5 & -\alpha_2 \end{pmatrix} , \quad (A.2)$$

$$a_{i;j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ -\beta_1 & -\beta_2 & -\beta_3 & -\beta_4 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \end{pmatrix} , \quad (A.3)$$

$$e_{i;j} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \alpha_3 & 0 & \alpha_2 & -\alpha_5 \\ 0 & 0 & 0 & 0 \\ \beta_5 & -\alpha_5 & 2\alpha_4 & -2\alpha_3 \end{pmatrix} , \quad (A.4)$$

$$h_{i;j} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \alpha_4 & 0 & \alpha_5 & \alpha_2 \\ -\beta_5 & \alpha_5 & -2\alpha_4 & 2\alpha_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (A.5)$$

If one introduces the vectors

$$\alpha_i \equiv \alpha_1 w_i + \alpha_2 a_i + \alpha_3 e_i + \alpha_4 h_i \quad , \tag{A.6}$$

$$\beta_i \equiv \beta_1 w_i + \beta_2 a_i + \beta_3 e_i + \beta_4 h_i \quad , \tag{A.7}$$

$$\gamma_i \equiv \gamma_1 w_i + \gamma_2 a_i + \gamma_3 e_i + \gamma_4 h_i \quad , \tag{A.8}$$

$$\xi_i \equiv -\alpha_3 w_i - \alpha_2 e_i + \alpha_5 h_i \quad , \tag{A.9}$$

$$\eta_i \equiv -\alpha_4 w_i - \alpha_5 e_i - \alpha_2 h_i \quad , \tag{A.10}$$

$$\zeta_i \equiv \beta_5 w_i - \alpha_5 a_i + 2\alpha_4 e_i - 2\alpha_3 h_i \quad , \tag{A.11}$$

then equations (A.2) - (A.5) may be expressed covariantly as

$$w_{i,j} = w_i \alpha_j + e_i \xi_j + h_i \eta_j \quad , \tag{A.12}$$

$$a_{i;j} = - a_i \alpha_j - e_i \beta_j - h_i \gamma_j \quad , \tag{A.13}$$

$$e_{i,j} = w_i \beta_j - a_i \xi_j + h_i \zeta_j \quad , \tag{A.14}$$

$$h_{i,j} = w_i \gamma_j - a_i \eta_j - e_i \zeta_j \quad . \tag{A.15}$$

The substitution of (A.12) - (A.15) into equations (3.19) - (3.22) yields

$$R_{2ijk} = w_i \alpha_{jk} + e_i \xi_{jk} + h_i \eta_{jk} \quad , \tag{A.16}$$

$$R_{1ijk} = -a_i\,\alpha_{jk} - e_i\,\beta_{jk} - h_i\,\gamma_{jk}\ , \tag{A.17}$$

$$R_{3ijk} = w_i\,\beta_{jk} - a_i\,\xi_{jk} + h_i\,\zeta_{jk}\ , \tag{A.18}$$

$$R_{4ijk} = w_i\,\gamma_{jk} - a_i\,\eta_{jk} - e_i\,\zeta_{jk}\ , \tag{A.19}$$

where

$$\alpha_{jk} \equiv \alpha_{j,k} - \alpha_{k,j} + \xi_j\,\beta_k - \xi_k\,\beta_j + \eta_j\,\gamma_k - \eta_k\,\gamma_j\ , \tag{A.20}$$

$$\beta_{jk} \equiv \beta_{j,k} - \beta_{k,j} + \beta_j\,\alpha_k - \beta_k\,\alpha_j + \zeta_j\,\gamma_k - \zeta_k\,\gamma_j\ , \tag{A.21}$$

$$\gamma_{jk} \equiv \gamma_{j,k} - \gamma_{k,j} + \gamma_j\,\alpha_k - \gamma_k\,\alpha_j - \zeta_j\,\beta_k + \zeta_k\,\beta_j\ , \tag{A.22}$$

$$\xi_{jk} \equiv \xi_{j,k} - \xi_{k,j} + \alpha_j\,\xi_k - \alpha_k\,\xi_j - \eta_j\,\zeta_k + \eta_k\,\zeta_j\ , \tag{A.23}$$

$$\eta_{jk} \equiv \eta_{j,k} - \eta_{k,j} + \alpha_j\,\eta_k - \alpha_k\,\eta_j + \xi_j\,\zeta_k - \xi_k\,\zeta_j\ , \tag{A.24}$$

$$\zeta_{jk} \equiv \zeta_{j,k} - \zeta_{k,j} + \beta_j\,\eta_k - \beta_k\,\eta_j + \xi_j\,\gamma_k - \xi_k\,\gamma_j\ . \tag{A.25}$$

If equations (A.16) - (A.19) are evaluated at P we obtain

$$R_{2112} = \alpha_{1,2} - \alpha_{2,1} + 2\alpha_1\,\alpha_2 - \alpha_3^2 - \alpha_4^2\ , \tag{A.26}$$

$$R_{2113} = \alpha_{1,3} - \alpha_{3,1} + 2\alpha_1\,\alpha_3 + 2\alpha_2\,\beta_1 + \alpha_4\,\beta_5 + \alpha_5\,\gamma_1\ , \tag{A.27}$$

$$\begin{aligned} R_{1321} = \beta_{1,2} - \beta_{2,1} &+ 2\beta_1\,\alpha_2 + \beta_3(\beta_2-\alpha_3) \\ &+ \beta_4(\gamma_2-\alpha_4) + \beta_5\,\gamma_2 + \alpha_5\,\gamma_1\ , \end{aligned} \tag{A.28}$$

$$R_{2114} = \alpha_{1,4} - \alpha_{4,1} + 2\alpha_1 \alpha_4 + 2\alpha_2 \gamma_1 - \alpha_3 \beta_5 - \alpha_5 \beta_1 \quad , \tag{A.29}$$

$$\begin{aligned} R_{1421} = \gamma_{1,2} - \gamma_{2,1} + 2\gamma_1 \alpha_2 + \gamma_3 (\beta_2 - \alpha_3) \\ + \gamma_4 (\gamma_2 - \alpha_4) - \beta_5 \beta_2 - \alpha_5 \beta_1 \quad , \end{aligned} \tag{A.30}$$

$$R_{2123} = \alpha_{2,3} - \alpha_{3,2} + 2\alpha_2 \beta_2 + \alpha_5 \gamma_2 \quad , \tag{A.31}$$

$$R_{2321} = \alpha_{3,2} - \alpha_2(\alpha_3 - \beta_2) - \alpha_5(\gamma_2 - 2\alpha_4) \quad , \tag{A.32}$$

$$R_{2124} = \alpha_{2,4} - \alpha_{4,2} + 2\alpha_2 \gamma_2 - \alpha_5 \beta_2 \quad , \tag{A.33}$$

$$R_{2421} = \alpha_{4,2} - \alpha_2(\alpha_4 - \gamma_2) + \alpha_5(\beta_2 - 2\alpha_3) \quad , \tag{A.34}$$

$$R_{2134} = \alpha_{3,4} - \alpha_{4,3} + \alpha_5(2\alpha_1 - \beta_3 - \gamma_4) + 2\alpha_2(\gamma_3 - \beta_4) \quad , \tag{A.35}$$

$$R_{3421} = -\beta_{5,2} - \alpha_{5,1} - 3\alpha_4 \beta_2 + 3\alpha_3 \gamma_2 + \alpha_1 \alpha_5 - \beta_5 \alpha_2 \quad , \tag{A.36}$$

$$\begin{aligned} R_{3113} = \beta_{1,3} - \beta_{3,1} + 3\beta_1 \alpha_3 + \beta_1 \beta_2 - \beta_3(\alpha_1 - \beta_3) \\ + \beta_4(\gamma_3 + \beta_5) + \beta_5 \gamma_3 - 2\alpha_4 \gamma_1 \quad , \end{aligned} \tag{A.37}$$

$$\begin{aligned} R_{3114} = \beta_{1,4} - \beta_{4,1} + 3\beta_1 \alpha_4 + \beta_2 \gamma_1 + \beta_4(\beta_3 - \alpha_1) \\ - \beta_5(\beta_3 - \gamma_4) + \beta_4 \gamma_4 + 2\alpha_3 \gamma_1 \quad , \end{aligned} \tag{A.38}$$

$$\begin{aligned} R_{1431} = \gamma_{1,3} - \gamma_{3,1} + 3\gamma_1 \alpha_3 + \gamma_2 \beta_1 - \gamma_3(\alpha_1 - \beta_3 - \gamma_4) \\ + \beta_5(\gamma_4 - \beta_3) + 2\alpha_4 \beta_1 \quad , \end{aligned} \tag{A.39}$$

$$R_{3123} = \beta_{2,3} - \beta_{3,2} + \beta_2^2 - \alpha_5\,\gamma_3 - 2\alpha_4\,\gamma_2 \quad , \tag{A.40}$$

$$R_{2331} = \alpha_{3,3} - \alpha_{2,1} + \alpha_3^2 + \alpha_2(\alpha_1+\beta_3) - 2\alpha_4^2 - \alpha_5\,\gamma_3 \quad , \tag{A.41}$$

$$R_{3124} = \beta_{2,4} - \beta_{4,2} + \beta_2\,\gamma_2 - \alpha_5\,\gamma_4 + 2\alpha_3\,\gamma_2 \quad , \tag{A.42}$$

$$R_{2431} = \alpha_{4,3} - \alpha_{5,1} + 3\alpha_3\,\alpha_4 + \alpha_5(\alpha_1+\beta_3) + \alpha_2\,\gamma_3 \quad , \tag{A.43}$$

$$\begin{aligned} R_{3134} = \beta_{3,4} - \beta_{4,3} + 2\beta_1\,\alpha_5 + \beta_2(\gamma_3-\beta_4) \\ -\alpha_4(\beta_3-2\gamma_4) + \alpha_3(\beta_4+2\gamma_3) \quad , \end{aligned} \tag{A.44}$$

$$R_{3431} = 2\alpha_{4,1} - \beta_{5,3} + 2\alpha_5\,\beta_1 - 3\alpha_4\,\beta_3 + 3\alpha_3\,\gamma_3 - \gamma_1\,\alpha_2 \quad , \tag{A.45}$$

$$\begin{aligned} R_{4114} = \gamma_{1,4} - \gamma_{4,1} + 3\gamma_1\,\alpha_4 + \gamma_1\,\gamma_2 + \gamma_4(\gamma_4-\alpha_1) \\ + \gamma_3(\beta_4-\beta_5) - \beta_5\,\beta_4 - 2\alpha_3\,\beta_1 \quad , \end{aligned} \tag{A.46}$$

$$R_{4123} = \gamma_{2,3} - \gamma_{3,2} + \beta_2\,\gamma_2 + \alpha_5\,\beta_3 + 2\alpha_4\,\beta_2 \quad , \tag{A.47}$$

$$R_{2341} = \alpha_{3,4} + \alpha_{5,1} + 3\alpha_3\,\alpha_4 - \alpha_5(\alpha_1+\gamma_4) + \alpha_2\,\beta_4 \quad , \tag{A.48}$$

$$R_{4124} = \gamma_{2,4} - \gamma_{4,2} + \gamma_2^2 + \alpha_5\,\beta_4 - 2\alpha_3\,\beta_2 \quad , \tag{A.49}$$

$$R_{2441} = \alpha_{4,4} - \alpha_{2,1} - 2\alpha_3^2 + \alpha_4^2 + \alpha_2(\alpha_1+\gamma_4) + \alpha_5\,\beta_4 \quad , \tag{A.50}$$

$$\begin{aligned} R_{4134} = \gamma_{3,4} - \gamma_{4,3} + 2\gamma_1\,\alpha_5 + \gamma_2(\gamma_3-\beta_4) \\ -\alpha_4(\gamma_3+2\beta_4) + \alpha_3(\gamma_4-2\beta_3) \quad , \end{aligned} \tag{A.51}$$

$$R_{3441} = -\beta_{5,4} - 2\alpha_{3,1} - 3\alpha_3\beta_4 + 3\alpha_3\gamma_4 + 2\gamma_1\alpha_5 + \alpha_2\beta_1 \quad , \tag{A.52}$$

$$R_{2323} = \alpha_{2,2} - 2\alpha_2^2 + \alpha_5^2 \quad , \tag{A.53}$$

$$R_{2324} \equiv -R_{2423} = -\alpha_{5,2} + 3\alpha_2\alpha_5 \quad , \tag{A.54}$$

$$R_{2334} = -\alpha_{2,4} - \alpha_{5,3} + \alpha_2\alpha_4 - \alpha_5\alpha_3 \quad , \tag{A.55}$$

$$R_{3423} = -\alpha_{5,3} - 2\alpha_{4,2} + \alpha_2(2\alpha_4+\gamma_2) - \alpha_5(2\beta_2-\alpha_3) \quad , \tag{A.56}$$

$$R_{2424} = \alpha_{2,2} - 2\alpha_2^2 + \alpha_5^2 \quad , \tag{A.57}$$

$$R_{2434} = \alpha_{2,3} - \alpha_{5,4} - \alpha_2\alpha_3 - \alpha_4\alpha_5 \quad , \tag{A.58}$$

$$R_{3424} = -\alpha_{5,4} + 2\alpha_{3,2} - \alpha_2(\beta_2+2\alpha_3) - \alpha_5(2\gamma_2-\alpha_4) \quad , \tag{A.59}$$

$$\begin{aligned} R_{3434} = {} & 2[\alpha_{4,4} + \alpha_{3,3} - 2\alpha_4^2 - 2\alpha_3^2] \\ & + 2\alpha_5(\beta_4-\gamma_3+\beta_5) - \alpha_2(\beta_3+\gamma_4) \quad . \end{aligned} \tag{A.60}$$

Part B

If the metric tensor is as in (4.16), then from (4.19), (4.20a), (4.20b) and (4.22) it follows that

$$\left.\begin{aligned} R_{11} = & -2\tau_{,11} - \left\{{1 \atop 11}\right\}_{,1} - \left\{{2 \atop 11}\right\}_{,2} - \left\{{3 \atop 11}\right\}_{,3} - \left\{{4 \atop 11}\right\}_{,4} \\ & + \left[\left\{{1 \atop 11}\right\} + 2\tau_{,1}\right]\left\{{1 \atop 11}\right\} + 2\left[\left\{{1 \atop 13}\right\} + \tau_{,3}\right]\left\{{3 \atop 11}\right\} + 2\left[\left\{{1 \atop 14}\right\} + \tau_{,4}\right]\left\{{4 \atop 11}\right\} \\ & + \left\{{2 \atop 12}\right\}^2 + \left\{{3 \atop 13}\right\}^2 + \left\{{4 \atop 14}\right\}^2 + 2\left\{{2 \atop 13}\right\}\left\{{3 \atop 21}\right\} \\ & + 2\left\{{2 \atop 14}\right\}\left\{{4 \atop 12}\right\} + 2\left\{{4 \atop 13}\right\}\left\{{3 \atop 14}\right\} \; . \end{aligned}\right\} \quad (B.1)$$

Since τ is independent of x^2 and

$$\beta = \xi x^2 + \xi_0 \; , \qquad (B.2)$$

$$\gamma = \eta x^2 + \eta_0 \; , \qquad (B.3)$$

$$\alpha = \tfrac{1}{2} \alpha_{,22}(x^2)^2 + \alpha_2 x^2 + \alpha_0 \; , \qquad (B.4)$$

where $\alpha_0, \alpha_2, \alpha_{,22}, \xi, \eta, \xi_0$ and η_0 are independent of x^2, it follows from the form of the Christoffel symbols that

$$R_{11} = R_{11}^{(2)}(x^2)^2 + R_{11}^{(1)} x^2 + R_{11}^{(0)} \; , \qquad (B.5)$$

where $R_{11}^{(2)}$, $R_{11}^{(1)}$ and $R_{11}^{(0)}$ are independent of x^2. Also, straightforward computations show that

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$$\left.\begin{aligned} R_{13} = {} & -\tau_{,13} + \tfrac{1}{2}\beta_{,21} - \alpha_{,32} - \beta_{,2}\tau_{,1} \\ & + \tfrac{1}{2}e^{2\tau}[\beta(\beta_{,2})^2 + \gamma\beta_{,2}\gamma_{,2} + \gamma_{,2}\gamma_{,3} + \\ & + \gamma(\gamma_{,3}-\beta_{,4})_{,2} - \beta_{,2}\gamma_{,4} - 2\tau_{,4}(\gamma\beta_{,2}-\beta\gamma_{,2}+\gamma_{,3}-\beta_{,4}) \\ & - \beta\beta_{,23} - \gamma\beta_{,24} + \beta_{,44} - \gamma_{,34}] \ , \end{aligned}\right\} \quad (B.6)$$

$$\left.\begin{aligned} R_{14} = {} & -\tau_{,14} + \tfrac{1}{2}\gamma_{,21} - \alpha_{,42} - \gamma_{,2}\tau_{,1} + \\ & + \tfrac{1}{2}e^{2\tau}[\gamma(\gamma_{,2})^2 + \beta\beta_{,2}\gamma_{,2} + \beta_{,2}\beta_{,4} + \beta(\beta_{,4}-\gamma_{,3})_{,2} \\ & - \gamma_{,2}\beta_{,3} - 2\tau_{,3}(\beta\gamma_{,2} - \gamma\beta_{,2} + \beta_{,4} - \gamma_{,3}) \\ & - \gamma\gamma_{,24} - \beta\gamma_{,23} + \gamma_{,33} - \beta_{,43}] \ . \end{aligned}\right\} \quad (B.7)$$

From (B.2) - (B.4) it is easily seen that

$$R_{13} = R^{(1)}_{13}\,x^2 + R^{0}_{13} \ , \qquad R_{14} = R^{(1)}_{14}\,x^2 + R^{0}_{14} \ , \qquad (B.8)$$

where $R^{(1)}_{13}$, $R^{(1)}_{14}$, $R^{(0)}_{13}$ and $R^{(0)}_{14}$ are independent of x^2 .